FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 3 Round 1
Arithmetic: Scientific & Base Notation

1.) {7,8,9}

2.) {14,16,13}

3.) {110,1111,202}

Note: Solutions are for form A only. All forms have similar solution methods.

1.) If $\frac{(10^{x+2})(100^{x-1})}{\left(\frac{1}{1000}\right)^{x}} = 10^{\{42,48,54\}}$, find the value of x.

Rewriting the left side as $(10^{x+2})(10^{2x-2})(10^{3x}) = 10^{42}$, we can simplify to $10^{6x} = 10^{42}$, so $x = 7$.

2.) Let a be a positive integer, and suppose that $101_{a+1} - 21_{a} = \{170_{10}, 226_{10}, 145_{10}\}$. What is 11_{a} expressed in base 10?

Rewriting the left side as an algebraic expression in terms of a, we get $(a+1)^2 + 1 - (2a+1) = 170$, which upon expanding the square gives $a^2 + 2a + 1 + 1 - (2a+1) = 170$, so $a^2 = 169$, so $a = 13$. This means $11_{13} = 1(13) + 1 = 14$.

3.) Hexadecimal (base 16) notation has 16 digits: 0-9, as well as A = 10, B = 11, C = 12, D = 13, E = 14, and F = 15. If $\{10,40,20\}, x, y,$ {6.C,F.A,7.D} are four consecutive terms of a geometric sequence in hexadecimal, find the value of x. Enter it as a number in base 3 notation.

In base 10, $10_{16} = 16$ and $6.\mathrm{C}_{16} = 6\frac{12}{16} = \frac{27}{4}$. This means that $16r^3 = \frac{27}{4}$, so $r = \frac{3}{4}$. Therefore $16\left(\frac{3}{4}\right) = 12$, which in base 3 is 110.

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Match 3 Round 2
Algebra 1: Word Problems

1.) $\{60,50,70\}$

2.) $\{68,77,86\}$

3.) $\{24,36,28\}$

Note: Solutions are for form A only. All forms have similar solution methods.

1.) Ivanna Hybrid commutes to work from home every day. On Monday her average speed in the morning is 50 miles per hour, but on the way home due to traffic she only manages an average speed of 40 miles per hour. If her total commute time on Monday was $\{\text{two hours and forty-two minutes, two hours and fifteen minutes, three hours and nine minutes}\}$, find the distance from her home to her work in miles.

Let D represent Ivanna's distance from home to work in miles. Since her total time in hours is $2\frac{42}{60} = \frac{27}{10}$, we can set up the equation $\frac{D}{50} + \frac{D}{40} = \frac{27}{10}$, which after multiplying through by 200 becomes $40D + 50D = 540$, or $90D = 540$, so $D = 60$.

2.) The formula for converting a temperature in degrees Celsius (C) to its exact equivalent in degrees Fahrenheit (F), is $F = \frac{9}{5}C + 32$. A popular approximate way of performing the same conversion is phrased as "double it and add 30". If on a given day the approximation gives a temperature in degrees Fahrenheit that is $\{2,3,4\}$ degrees Fahrenheit higher than the actual temperature, find the actual temperature in degrees Fahrenheit.

Let x represent the actual temperature in degrees Celsius. From the problem we can set up $2x + 30 = 2 + \frac{9}{5}x + 32$, which gives us $\frac{1}{5}x = 4$, so $x = 20$. Since we need temperature in degrees Fahrenheit, we get $\frac{9}{5}(20) + 32 = 68$.

3.) A large bowl contains only peanuts and cashews. Initially, the bowl contains a total of {30,45,35} cups of nuts. Then {6,4,4} cups of peanuts and {9,11,9} cups of cashews are added to the bowl. As a result, the quantity $\frac{\text{number of cups of peanuts}}{\text{number of cups of cashews}}$ is now one half of what it was previously. How many cups of peanuts were in the bowl originally?

Let x equal to the original number of cups of peanuts in the bowl. From the problem we can set up $\frac{x+6}{30-x+9} = \frac{1}{2} * \frac{x}{30-x}$. Multiplying through by the least common denominator gives $2(x + 6)(30 - x) = x(39 - x)$, which expands to make $360 + 48x - 2x^2 = 39x - x^2$, which written in standard form gives $x^2 - 9x - 360 = 0$. This factors into $(x - 24)(x + 15) = 0$, so $x = 24$.

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Match 3 Round 3 Geometry: Polygons

1.) {54,65,77}

2.) {41,73,124}

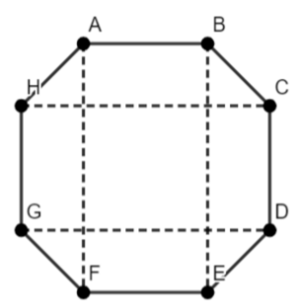
3.) {3540,1980,1560}

Note: Solutions are for form A only. All forms have similar solution methods.

1.) How many diagonals are in a polygon whose interior angle measures in degrees total {1800,1980,2160}?

We can solve for the number of sides using $180(n-2) = 1800$, so $n = 12$. From here we evaluate $\frac{12(12-3)}{2}$ to get 54.

2.) The diagram (not drawn to scale) shows octagon $ABCDEFGH$. $AB = CD = EF = GH = 1$ and $BC = DE = FG = HA = x\sqrt{2}$. Each interior angle of the octagon measures 135°. The area of the octagon is {3,5,9}. Find $100x$ rounded to the nearest integer.



From the description and diagram we can see that the area of the octagon can be divided into four isosceles right triangles, four rectangles and one square. Each isosceles right triangle has legs of length x, making a total area of $2x^2$. Each rectangle has sides of length 1 and x, making the total area $4x$. Finally the square is a unit square, and setting the total area equal to 3 gives $2x^2 + 4x + 1 = 3$. Solving for x with the quadratic formula gives $x = \frac{-4 \pm \sqrt{32}}{4}$. The simplified positive result is $\sqrt{2} - 1$, so $100x$ rounded to the nearest integer is 41.

3.) A regular m-gon and a regular n-gon, with $m < n$, have the property that the difference between the measure of one interior angle of each is {6,8,9} degrees. Find the largest possible value of n.

Noting that the n-gon will have larger angles, we can set up $180 - \frac{360}{n} - \left(180 - \frac{360}{m}\right) = 6$, which simplifies to $\frac{360}{m} - \frac{360}{n} = 6$. Dividing both sides by 360 and combing the fractions gives $\frac{n-m}{mn} = \frac{1}{60}$. We can then solve for n by cross multiplication, giving $60n - mn = 60m$, so $n = \frac{60m}{60-m}$. We can see that the value of m that will produce the largest positive integer value of n is 59, which gives $n = 60(59) = 3540$.

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Match 3 Round 4
Algebra 2: Functions & Inverses

1.) {27,31,35}

2.) {6,10,7}

3.) {35,29,23}

Note: Solutions are for form A only. All forms have similar solution methods.

Note: the inverse f^{-1} of a function f is not necessarily a function.

1.) If $f(x) = 2x + 5$ and $f^{-1}(f^{-1}(a)) = \{3,4,5\}$, find the value of a.

Since $f^{-1}(f^{-1}(a)) = 3$, we know $a = f(f(3)) = f(11) = 27$.

2.) Let k be a rational number that can be expressed in simplest form as $\frac{a}{b}$ where a and b are integers. If $g(x) = x^3 + 3x - 10$ and $g^{-1}(\{10k^2 - 6k + 5, 14k^2 + 2k - 3, 36k^2 + 2k + 8\}) = 2k$, find $\frac{1}{2}ab - b$.

Since we know $g(2k) = 10k^2 - 6k + 5$, this means $(2k)^3 + 3(2k) - 10 = 10k^2 - 6k + 5$, which in standard form gives $8k^3 - 10k^2 + 12k - 15 = 0$. This factors by grouping into $(2k^2 + 3)(4k - 5) = 0$. This has one rational solution for k of $\frac{5}{4}$. Therefore $\frac{1}{2}(5)(4) - 4 = 6$.

3.) Let $f(x) = a\sqrt{x + b} + c$, where a, b, and c are real constants. If $2f(4x + 1)$ has a domain of $x \geq -\frac{9}{16}$ and a range of $y \leq 100$ and $f(1) = 41$, find $\{f(5), f(11), f(19)\}$.

Note that the range of $f(4x+1)$ is $y \leq 50$. This means that $a < 0$ and $c = 50$. Since $f(4x+1) = a\sqrt{4x+1+b} + 50$, which we can rewrite as $f(4x+1) = 2a\sqrt{x + \frac{1+b}{4}} + 50$, we can use the given domain to see that $\frac{1+b}{4} = \frac{9}{16}$, which we can solve to get $b = \frac{5}{4}$. Finally we can solve for a using $a\sqrt{1 + \frac{5}{4}} + 50 = 41$, so $\frac{3}{2}a + 50 = 41$, which means $a = -6$. Knowing all required values, we find $f(5) = -6\sqrt{5 + \frac{5}{4}} + 50 = -6\left(\frac{5}{2}\right) + 50 = 35$.

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Match 3 Round 5
Precalculus: Exponents & Logarithms

1.) $\{616,216,416\}$

2.) $\{81,625,2401\}$

3.) $\{21,29,25\}$

Note: Solutions are for form A only. All forms have similar solution methods.

1.) If a and b are positive integers and $\frac{5^a * 2^b}{5^{10} * 2^3} = 8 * 10^{\{300,100,200\}}$, find the value of $a + b$.

We can rewrite the equation as $5^{a-10} * 2^{b-3} = 2^3 * 2^{300} * 5^{300}$. This means $a - 10 = 300$ and $b - 3 = 303$, so $a = 310$ and $b = 306$, so $a + b = 616$.

2.) If $\log_{\{3,5,7\}}\left(1 - \frac{1}{2}\right) + \log_{\{3,5,7\}}\left(1 - \frac{1}{3}\right) + \cdots + \log_{\{3,5,7\}}\left(1 - \frac{1}{n}\right) = -4$, find the value of n.

Rewriting the expressions in the logarithms and combining gives $\log_3(\frac{1}{2} * \frac{2}{3} * \frac{3}{4} * \ldots * \frac{n-2}{n-1} * \frac{n-1}{n})$, which simplifies to $\log_3\left(\frac{1}{n}\right) = -4$. This means $3^{-4} = \frac{1}{n}$, so $n = 81$.

3.) If $\log_3\{45,135,405\} = x$ and $\log_4\{400,1600,100\} = y$, then $\log_3 4 = \frac{ax-b}{cy-d}$ for positive integers $a, b, c,$ and d, where a and c are relatively prime. Find the value of $a + 2b + 3c + 4d$.

We can rewrite $\log_3 45$ as $\log_3(3^2 * 5)$, so $2 + \log_3 5 = x$. Similarly, we can rewrite $\log_4 400$ as $\log_4(4^2 * 5^2)$ so $2 + 2\log_4 5 = y$. This means $\frac{\log 5}{\log 3} = x - 2$ and $\frac{\log 5}{\log 4} = \frac{1}{2}y - 1$. Therefore $\log_3 4 = \frac{\log 4}{\log 3} = \frac{\frac{\log 5}{\log 3}}{\frac{\log 5}{\log 4}} = \frac{x-2}{\frac{1}{2}y-1} = \frac{2x-4}{y-2}$, so $a + 2b + 3c + 4d = 2 + 2(4) + 3(1) + 4(2) = 21.$

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Match 3 Round 6
Miscellaneous: Matrices

1.) $\{22,17,12\}$

2.) $\{6,8,10\}$

3.) $\{43,45,48\}$

Note: Solutions are for form A only. All forms have similar solution methods.

1.) If $\begin{bmatrix} a & a \\ c & c \end{bmatrix} + \{2,3,4\}\begin{bmatrix} 6 & b \\ b & 1 \end{bmatrix} = \begin{bmatrix} 15 & 33 \\ 50 & d \end{bmatrix}$, find the value of d.

The equation gives $a + 12 = 15$, $a + 2b = 33$, $c + 2b = 50$, and $c + 2 = d$, so $a = 3, b = 15, c = 20$, and $d = 22$.

2.) If $A = \begin{bmatrix} 4 & 2 \\ 7 & x \end{bmatrix}$ and $\det(2A^{-1}) = \left\{\frac{2}{5}, \frac{2}{9}, \frac{2}{13}\right\}$, find the value of x.

One way to solve this is to find $2A^{-1}$ in terms of x, giving $\frac{1}{4x-14}\begin{bmatrix} 2x & -4 \\ -14 & 8 \end{bmatrix}$, so $\frac{(2x)(8)-(-4)(-14)}{(4x-14)^2} = \frac{16x-56}{(4x-14)^2} = \frac{4}{4x-14} = \frac{2}{5}$, giving $x = 6$. Another way is to use determinant properties to note that $\det(A^{-1}) = \left(\frac{1}{2^2}\right)\frac{2}{5} = \frac{1}{10}$, s o $\det(A) = 10$, and $4x - 14 = 10$ gives $x = 6$.

3.) If $= \begin{bmatrix} x & 8 \\ y & z \end{bmatrix}, B = \begin{bmatrix} 9 & -1 \\ 3 & 4 \end{bmatrix}, x > 0$, $AB = BA$, and $\det(A) = \{321, 417,576\}$, find the value of z.

Since $AB = \begin{bmatrix} 9x + 24 & -x + 32 \\ 9y + 3z & -y + 4z \end{bmatrix}$ and $BA = \begin{bmatrix} 9x - y & 72 - z \\ 3x + 4y & 24 + 4z \end{bmatrix}$, we can see immediately that $y = -24$. We also see that $-x + 32 = 72 - z$, so $x =$

$z - 40$. Therefore, $\det(A) = z(z - 40) - 8(-24) = 321$, which in standard form gives $z^2 - 40z - 129 = 0$, which factors into $(z - 43)(z + 3) = 0$. Since $x > 0$ and $x = z - 40$, the only solution that works is $z = 43$.

Team Round

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1.) 80
2.) 108
3.) 45
4.) 10201
5.) 11100
6.) 4

1.) Mike and Andrew rake leaves. It would take Andrew twenty minutes longer to rake a particular lawn than it would take Mike. Mike starts raking alone. After twenty minutes Andrew joins him, but after another twenty minutes Andrew gets bored and quits. It takes Mike another 24 minutes to finish raking the lawn alone. Assuming that each person's raking did not interfere with the other's, how many minutes would it have taken Mike to rake the lawn alone?

Let M be the number of minutes it would take Mike to rake the lawn alone and A be the number of minutes it would take Andrew to rake the lawn alone. We know $A = M + 20$. From the problem we can set up $20\left(\frac{1}{M}\right) + 20\left(\frac{1}{M} + \frac{1}{A}\right) + 24\left(\frac{1}{M}\right) = 1$. Substituting and combining gives $\frac{64}{M} + \frac{20}{M+20} = 1$. Multiplying by the least common denominator gives $64(M + 20) + 20M = M(M + 20)$, which in standard form gives $M^2 - 64M - 1280 = 0$, which factors into $(M - 80)(M + 16) = 0$, so $M = 80$.

2.) For a particular value of k, the function $f(x) = \frac{\sqrt{x}-k}{x^2-9}$ has a range of $(0, a) \cup (a, b]$, where a and b are positive real numbers. Find the value of $\frac{1}{ab}$.

Since the domain of the function contains no negative numbers, the only discontinuity will be at $x = 3$. This discontinuity will be an asymptote and lead to an unbounded range unless it is the location of a removable discontinuity (hole). This will occur when $\sqrt{x} - k$ matches a factor of the bottom. Rewriting as $f(x) = \frac{\sqrt{x}-k}{(x+3)(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}$ reveals that $k = \sqrt{3}$, and besides at $x = 3$, the function's behavior can be modeled with $f(x) = \frac{1}{(x+3)(\sqrt{x}+\sqrt{3})}$. Therefore, the largest range value will occur when $x = 0$, so $b = \frac{1}{3\sqrt{3}}$, and a is the y-value of the removable discontinuity can be found by substituting $x = 3$ in the reduced form to get $a = \frac{1}{12\sqrt{3}}$. Therefore $\frac{1}{ab} = 108$.

3.) Matrix A has the property that the entries in each row form a unique arithmetic sequence. If $B = \begin{bmatrix} 3 & 1 \\ 6 & -4 \\ 2 & 3 \end{bmatrix}$ and $AB = \begin{bmatrix} 52 & 6 \\ 112 & -4 \end{bmatrix}$, find the sum of the elements of A.

Note that A must have dimensions 2 x 3 in order to produce a 2 x 2 product with a 3 x 2 matrix. The elements of the first row of A can be represented as , $x + y$, and $x + 2y$. Setting $3x + 6(x + y) + 2(x + 2y) = 52$ and $x - 4(x + y) + 3(x + 2y) = 6$ gives $x = 2$ and $y = 3$, making the elements of the first row 2, 5, and 8 respectively. The same process can be set up for the second row to find that the elements are 12, 10, and 8. Therefore the sum of the elements is 45.

4.) Assuming the pattern continues, find the following product of 100 logarithms: $\log_5(25^2)\log_{25^2}(125^3)\log_{125^3}(625^4)\dots$.

By extrapolating the pattern and using properties of logarithms, this product can be written as $\frac{\log 25^2}{\log 5} * \frac{\log 125^3}{\log 25^2} * \dots * \frac{\log((5^{101})^{101})}{\log((5^{100})^{100})}$, which simplifies to $\frac{\log((5^{101})^{101})}{\log 5} = \log_5((5^{101})^{101}) = (101)(101) = 10201$.

5.) The number 2020! has 5802 digits. If 2020! were written in scientific notation as $a * 10^b$ where a is a decimal between 1 and 10 with n digits (and no trailing zeros) and b is a positive integer, find the value of $n + b$.

We can see that $b = 5801$. To find n, we can find the number of trailing zeros and subtract them off. The number of zeros is equal to the number of tens in the factorization of 2020!, which, since 2 is a more common factor than 5, is equal to the number of fives in the factorization. This can be computed using $\text{floor}(\frac{2020}{5}) + \text{floor}(\frac{2020}{25}) + \text{floor}(\frac{2020}{125}) + \text{floor}(\frac{2020}{625})$, equaling $404 + 80 + 16 + 3 = 503$, so $n = 5802 - 503 = 5299$, so $n + b = 11100$.

6.) An n-gon has d diagonals, and, in a regular d-gon, each interior angle has an integer degree measure. How many different possible values of n are there?

Since $d = \frac{n(n-3)}{2}$, we can write an interior angle of a regular d-gon in terms of n as $180 - \frac{720}{n(n-3)}$. Therefore we need to find all values $n > 3$ such that this expression comes out to an integer. This produces $n \in \{4,5,6,8,15\}$. However, we cannot include $n = 4$ since a quadrilateral has only 2 diagonals. Therefore there are 4 possible values of n.