FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 1 Round 1
Arithmetic: Percents

1.) {901,1901,2901}

2.) {26,22,39}

3.) {10,6,42}

Note: Solutions below are completed for the values from Form A only. Solution methods are identical for all Forms.

1.) When the number *x* is increased by *x* percent, the result is {10,20,30} less than twice *x*. If that statement is represented by the equation $Ax^2 + Bx + C = 0$, where $A > 0$ and A, B, and C are integers with no common factors larger than 1, find the value of $A + B + C$.

Translating the statement into an equation yields $\left(1 + \frac{x}{100}\right)x = 2x - 10$, which when expanded and written in standard form yields $\frac{1}{100}x^2 - x + 10 = 0$, or $x^2 - 100x + 1000 = 0$, so $A + B + C = 1 + (-100) + 1000 = 901$.

2.) There exist specific values of *w* and *k* for which the following statement is true for all values of *x:* For constants *x*, *y*, and *z*, If {20,30,40} percent of *x* is {4,5,3} more than *y* and {60,40,70} percent of *y* is 1 less than *z*, then *k* percent of *x* is *w* more than *z*. Find the value of 10*w* + *k*.

As equations, the statements are $.2x = y + 4$ and $.6y = z - 1$. Solving for y in the first equation yields $y = .2x - 4$, and substituting for y in the second equation yields $.6(.2x - 4) = z - 1 \rightarrow .12x - 2.4 = z - 1 \rightarrow .12x = z + 1.4$, so $k = 12$ and $w = 1.4$, and $10w + k = 10(1.4) + 12 =$

26.

3.) The percent difference between p and q is defined as $\frac{|p-q|}{\frac{p+q}{2}} \times 100\%$. Two positive numbers m and n with $m > n$ have the following property: The percent difference between $2m$ and n is equal to $\left\{\frac{14}{9}, \frac{5}{2}, \frac{26}{5}\right\}$ times the percent difference between m and n. If the ratio of m to n can be expressed in simplest form by the fraction $\frac{a}{b}$, find ab.

Translated to an equation, this relationship can be written as $\frac{|2m-n|}{\frac{2m+n}{2}} * 100 = \frac{14|m-n|}{9\left(\frac{m+n}{2}\right)} * 100$. Since $m > n$, this can be simplified without the absolute value symbols to $\frac{2m-n}{2m+n} = \frac{14(m-n)}{9(m+n)}$, which means $(2m-n)(9m+9n) = (2m+n)(14m-14n)$, which expands to make $18m^2 + 9mn - 9n^2 = 28m^2 - 14mn - 14n^2 \rightarrow 10m^2 - 23mn - 5n^2 = 0$, which factors to make $(2m-5n)(5m+n) = 0$. Since m and n are both positive, their ratio must be positive, so $\frac{m}{n} = \frac{5}{2}$, and $a = 5$ and $b = 2$, so $ab = 10$.

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Match 1 Round 2 Algebra 1: Equations

1.) $\{17{,}25{,}13\}$

2.) $\{36{,}64{,}100\}$

3.) $\{289{,}252{,}404\}$

Note: Solutions below are completed for the values from Form A only. Solution methods are identical for all Forms.

1.) If $x = \{4,3,5\}$ is a solution to the equation $(862A)x - 45987 = 749A$ for some constant A, find the value of A rounded to the nearest integer.

Letting $x = 4$ gives the equation $3448A - 45987 = 749A$, which yields a value of A of $\frac{45987}{2669}$, which rounds to 17.

2.) Find the nonzero value of k such that the equation $(x + \{3,4,5\})(x^2 + k) = x^3$ has only one solution for x.

Expanding the product of binomials gives $x^3 + 3x^2 + kx + 3k = x^3$, which simplifies to $3x^2 + kx + 3k = 0$. This will have only one solution for x when $k^2 - 4(3)(3k) = 0$, which factors to make $k(k - 36) = 0$, so $k = 36$

3.) For how many integer pairs (a, b) is $\{4,5,2\}a - 3b = 1$ and $0 \le a + b \le 2020$?

We can notice by inspection that $(1{,}1)$ satisfies the equation and is the ordered pair whose sum is the least possible positive number. Since 4 and 3 are relatively prime, the next ordered pair will occur when a increases by 3 and b increases by 4, or (4,5). This means that for each ordered pair, the sum of a and b will increase by 7. If n is the number of ordered pairs, this problem can be modeled by the inequality $2 + 7(n - 1) \leq 2020$. This gives $n \leq \frac{2018}{7} + 1$, and the largest integer value of n that satisfies this inequality is $n = 289$.

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Match 1 Round 3 Geometry: Triangles & Quadrilaterals

1.) $\{22,20,30\}$

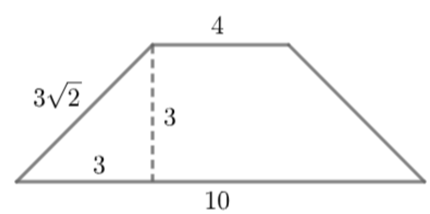
2.) $\{16,20,26\}$

3.) $\{10,12,8\}$

Note: Solutions below are completed for the values from Form A only. Solution methods are identical for all Forms.

1.) An isosceles trapezoid has an area of $\{21,14,40\}$, a height of $\{3,2,4\}$, and one base of length $\{10,9,14\}$. If the perimeter of the trapezoid is $a + b\sqrt{c}$ where a, b, and c are positive integers and c has no perfect square factors greater than 1, find $a + b + c$.

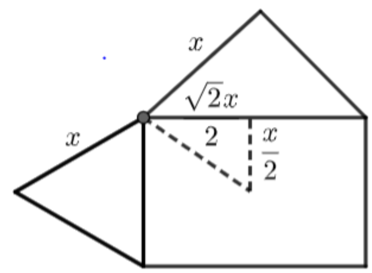
Since $21 = \frac{1}{2}(3)(10 + b_2)$, we know the second base has a length of 4. This gives us enough information to solve for the remaining side lengths (see the diagram). This means the perimeter is $14 + 6\sqrt{2}$, so $a + b + c = 22$.



2.) What is the perimeter of a rectangle with a diagonal of length $\{6,8,11\}$ and an area of $\{14,18,24\}$?

Let the lengths of the sides of the rectangle be x and y. From the diagonal, we know that $x^2 + y^2 = 36$, and from the area we know that $xy = 14$. Therefore $x^2 + 2xy + y^2 = 36 + 2(14) = 64 = (x + y)^2$, so $x + y = 8$. The perimeter is $2x + 2y$, or 16.

3.) A right rectangular pyramid has two lateral faces with a vertex angle of 90 degrees and two lateral faces with a vertex angle of 60 degrees. If the base of the pyramid has an area of $\{400\sqrt{2}, 576\sqrt{2}, 256\sqrt{2}\}$, find the height of the pyramid.



See the diagram of part of the net of the pyramid. The edges of the lateral faces must have equal lengths to form a pyramid, and we will call this length x. We can use the properties of special right triangles and Pythagorean theorem to solve for the distance from a base vertex to the center of the base in terms of x, which comes out to $\frac{\sqrt{3}}{2}x$. We can solve for the height of the pyramid h in terms of x using $h^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = x^2$, which gives $h = \frac{x}{2}$. Noting that the area of the base is $x^2\sqrt{2} = 400\sqrt{2}$, we find that $x = 20$ and therefore $h = 10$.

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Match 1 Round 4 Algebra 2: Simultaneous Equations

1.) _______{8,9,10}_____________

2.) ________{20,6,8}____________

3.) ________{48,12,24}________

Note: Solutions below are completed for the values from Form A only. Solution methods are identical for all Forms.

1.) For a concert, tickets cost \$68 for an adult and \$31 for a child. For a particular group of {12, 14, 16} people, the cost of the tickets is {\$668, \$767, \$866}. How many adults are in the group?

Letting x be the number of adults and y be the number of children, we get the system $\begin{cases} x+y=12 \\ 68x+31y=668 \end{cases}$. This can be solved manually by substituting $y=12-x$ into the second equation to give $68x+31(12-x)=668$, which gives $x=8$.

2.) The graphs of $y=4x-x^2$ and $y=kx^2$, where k is a positive constant, intersect at points M and N. If the slope between M and N is $\left\{\frac{3}{5},\frac{2}{3},\frac{1}{2}\right\}$, then the value of k can be written as $\frac{a}{b}$ where a and b are relatively prime integers and $b>0$. Find $a+b$.

We can see that since $k>0$, the one of the two points of intersection will be the origin (M) and the other will be in Quadrant I (N). Let the coordinates of N be (c,d). Solving for c gives the equation $4c-c^2=kc^2$, which gives $(k+1)c=4$, or $c=\frac{4}{k+1}$. We also know $d=kc^2=\frac{16k}{(k+1)^2}$. The slope between M and N is then $\frac{d}{c}$, which simplifies to $\frac{4k}{k+1}$. Setting $\frac{4k}{k+1} = \frac{3}{5}$ yields $k = \frac{3}{17}$, so $a = 3$ and $b = 17$, giving $a + b = 20$.

3.) The ordered pair $\left\{\left(2, \frac{17}{3}\right), \left(-1, \frac{5}{12}\right), \left(1, \frac{13}{6}\right)\right\}$ is one of infinite solutions of the system $\begin{cases} 4x - Ay = -9 \\ Bx + 2y = C \end{cases}$ for constants $A, B,$ and C. Find $|ABC|$.

We can substitute the point $\left(2, \frac{17}{3}\right)$ for x and y in the first equation to get $A = 3$. Since each term in the second equation must be a constant multiple of its corresponding term in the first equation and $2 = -\frac{2}{3}(3)$, we know $B = -\frac{2}{3}(4) = -\frac{8}{3}$ and $C = -\frac{2}{3}(-9) = 6$, so $|ABC| = \left|3\left(-\frac{8}{3}\right)(6)\right| = 48$.

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Match 1 Round 5 Precalculus: Right Triangle Trigonometry

1.) _____{19,57,23}______________

2.) ______{3,5,7}________________

3.) ______{4,5,6}________________

Note: Solutions below are completed for the values from Form A only. Solution methods are identical for all Forms.

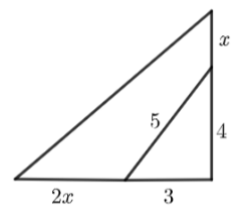
1.) In right triangle ABC with right angle C, if $\tan(A) = \{7,5,3\}$, then $\cos(B)$ can be expressed in simplest radical form as $\frac{x\sqrt{y}}{z}$ where x, y, and z are integers. Find $x + y + z$.

If $\tan(A) = 7$, then $\sin(A) = 7\cos\,(A)$. Given $\sin^2(A) + \cos^2(A) = 1$. This means $(7\cos(A))^2 + \cos^2(A) = 1$, or $\cos^2(A) = \frac{1}{50}$, and $\cos(A) = \frac{\sqrt{2}}{10}$. This means $\cos(B) = \sin(A) = 7\cos(A) = \frac{7\sqrt{2}}{10}$, and $x + y + z = 7 + 2 + 10 = 19$.

2.) Consider right triangle ABC with right angle A. If the hypotenuse has a length of $\{2\sqrt{5}, 2\sqrt{13}, 10\}$ units and the value of $\tan\,(B)$ has the same value as the area of the triangle in square units, find the area of the triangle in square units.

Given $\tan(B) = \frac{b}{c}$ and the area of the triangle is $\frac{1}{2}bc$, we get $\frac{b}{c} = \frac{1}{2}bc$, which means $c = \sqrt{2}$. We can find the length of b using Pythagorean theorem: $2 + b^2 = 20$, which gives $b = 3\sqrt{2}$. The area is then $\frac{1}{2}bc = 3$.

3.) You are standing on a straight road. You see a balloon being released from a point on the road, and a little later, at time t_1, the balloon has risen vertically, and the sine of the angle of elevation from the ground where you stand to the balloon is $\frac{4}{5}$. You run along the road in the direction away from the launch point and stop at time t_2, and find that the distance you ran is twice the height the balloon has climbed since t_1. The tangent of the new angle of elevation from the ground where you stand to the balloon is $\left\{\frac{16}{27}, \frac{4}{7}, \frac{24}{43}\right\}$. If the height of the balloon at t_1 is h_1 and the height of the balloon at t_2 is h_2, find $\frac{h_2}{h_1}$.



See the diagram (not drawn to scale). Without loss of generality, we will set the dimensions of the original triangle (with the first angle of elevation) to 3, 4, and 5. We can then set up $\frac{x+4}{2x+3} = \frac{16}{27}$, which can solve to give $x = 12$. We can then find $\frac{t_2}{t_1} = \frac{12}{4} = 3$.

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Match 1 Round 6 Miscellaneous: Coordinate Geometry

1.) _________{11,10,38}_________

2.) _________{9,12,15}__________

3.) _________{1875,450,2000}____

Note: Solutions below are completed for the values from Form A only. Solution methods are identical for all Forms.

1.) A straight line intersects the x-axis at $\{(4,0),(6,0),(10,0)\}$ and the y-axis at $\{(0,8),(0,2),(0,6)\}$. The equation of the line is $Ax + By = C$, where A, B, and C are integers, $A > 0$, and the only positive integer that divides all of A, B, C is 1. Find $A + B + C$.

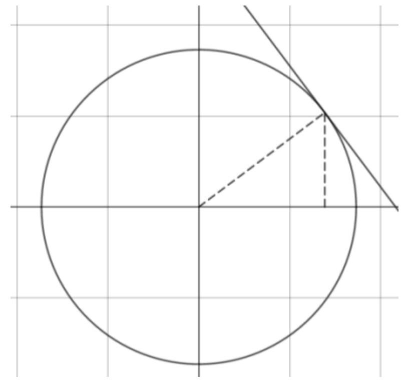
Given $4A = 8B = C$, the smallest positive integer values of A, B, and C that work are $A = 2$, $B = 1$, and $C = 8$, so $A + B + C = 11$.

2.) The point P with coordinates $\{(10,5),(11,7),(12,9)\}$ is reflected across the line $y = 2x$ to make the new point P'. Find the sum of the coordinates of P'.

Both P and P' must lie on the line $y = -\frac{1}{2}x + b$. Substituting the point (10,5) gives $b = 10$. The lines $y = 2x$ and $y = -\frac{1}{2}x + 10$ intersect at (4,8). Because this point is the midpoint of P and P', the coordinates of P' must be $(-2,11)$, and $-2 + 11 = 9$.

3.) A circle centered at the origin with an area of $\{75\pi, 18\pi, 80\pi\}$ is tangent to the line $4x + 3y = k$, where k is a constant. Find the v-alue of k^2.

See the diagram. Since the tangent line has a slope of $-\frac{4}{3}$, the radius it intersects must lie along the line $y = \frac{3}{4}x$. Substituting into the equation for the circle gives $x^2 + \frac{9}{16}x^2 = 75$, which gives $x^2 = 48$, so $x = 4\sqrt{3}$ and consequently $y = 3\sqrt{3}$. Substituting into the equation of the line gives $4(4\sqrt{3}) + 3(3\sqrt{3}) = 25\sqrt{3} = k$, so $k^2 = 1875$.

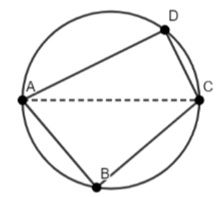


Team Round

FAIRFIELD COUNTY MATH LEAGUE 2020-2021 Match 1 Team Round

1.) 26
2.) 4
3.) 636
4.) 205
5.) 7
6.) 150

1.) Consider quadrilateral $ABCD$, inscribed in a circle, where diagonal $\overline{AC}$ is a diameter of the circle. If $\tan(\angle BAC) = \frac{4}{3}$ and $\tan(\angle CAD) = \frac{7}{24}$ and $AD = 8$, find the area of $ABCD$.

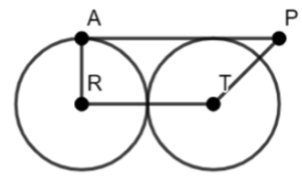


See the diagram (not drawn to scale). Because $\overline{AC}$ is the diameter of the circle, triangles ADC and ABC are right triangles with right angles D and B. Since $AD = 8$, we know $CD = \frac{7}{3}$ and $AC = \frac{25}{3}$. This also means that $AB = 5$ and $BC = \frac{20}{3}$. The area of the quadrilateral is the sum of the areas of the two right triangles: $\frac{1}{2}(8)\left(\frac{7}{3}\right) + \frac{1}{2}(5)\left(\frac{20}{3}\right) = 26$.

2.) The road from Ridgefield to Wilton is 5 miles uphill, then 4 miles on level ground, then 6 miles downhill. Mr. Corbishley has a consistent uphill walking speed, a consistent walking speed on level ground, and a consistent downhill walking speed. He walks from Wilton to Ridgefield in 4 hours. Later he walks the first half of the journey from Ridgefield to Wilton and returns to Ridgefield in a total of 3 hours and 55 minutes. Still later he walks from Ridgefield to Wilton in 3 hours and 52 minutes. Find Mr. Corbishley's walking speed on level ground in miles per hour. (Don't enter units.)

Let x represent Mr. Corbishley's uphill speed, y his level speed, and z his downhill speed, all in miles per minute. His first trip is 6 miles uphill, 4 miles level, and 5 miles downhill, giving $\frac{6}{x} + \frac{4}{y} + \frac{5}{z} = 240$. Half of the journey is 7.5 miles, so his second trip is 5 miles uphill, 2.5 miles level (twice), and 5 miles downhill, giving $\frac{5}{x} + \frac{5}{y} + \frac{5}{z} = 235$. His final journey is 5 miles uphill, 4 miles level, and 6 miles downhill, giving $\frac{5}{x} + \frac{4}{y} + \frac{6}{z} = 232$. Subtracting the first equation from the second gives $\frac{1}{x} - \frac{1}{y} = 5$, and subtracting six times the first equation from five times the third gives $\frac{11}{x} + \frac{4}{y} = 280$, which after subtracting eleven times the prior equation in x and y gives $\frac{15}{y} = 225$, so $y = \frac{1}{15}$ miles per minute, or 4 miles per hour.

3.) The diagram shows two circles, each with area 288π, which are tangent to each other. Trapezoid $TRAP$ is drawn so that points T and R are the centers of the circles, $\overline{AP}$ is tangent to both circles, A lies on $\odot R$, and $AP > TR$. If the perimeter of $TRAP$ is $78\sqrt{2}$, find the area of $TRAP$.



The area of the circles tell us that $AR = 12\sqrt{2}$ an $TR = 24\sqrt{2}$. Let $PT = x$. Using Pythagorean theorem, we can write the perimeter as $12\sqrt{2} + 24\sqrt{2} + 24\sqrt{2} + x + \sqrt{x^2 - 288}$. Setting this equal to $78\sqrt{2}$ gives us $18\sqrt{2} - x = \sqrt{x^2 - 288}$. Squaring both sides gives the equation $x^2 - 36\sqrt{2}x + 648 = x^2 - 288$, which means $36\sqrt{2}x = 936$, which gives $x = 13\sqrt{2}$ and $AP = 29\sqrt{2}$. This makes the area $\frac{1}{2}(12\sqrt{2})(24\sqrt{2} + 29\sqrt{2}) = 636$.

4.) A, B, and C are positive numbers. $C\%$ of B is 20 less than A. $(2A)\%$ of C is 17 more than twice B. $((2A)\%$ of $B)\%$ of $2C$ is 117. Find $A + B + C$.

Writing the equations as a system and simplifying gives: $\begin{cases} \frac{BC}{100} = A - 20 \\ \frac{AC}{50} = 2B + 17 \\ \frac{ABC}{2500} = 117 \end{cases}$. The third equation gives $ABC = 292500$. Multiplying the top equation by A gives $\frac{ABC}{100} = A^2 - 20A$, and substituting for ABC gives the quadratic $A^2 - 20A - 2925 = 0$, which has a positive solution of $A = 65$. Multiplying the second equation by B and substituting for ABC gives $2B^2 + 17B - 5850 = 0$, which has a positive solution of $B = 50$. Finally, solving for C in any equation gives $C = 90$, and $65 + 50 + 90 = 205$.

5.) Find the product of all values of a such that the equation $\frac{x-4}{2x-5} = \frac{x+a}{2x+7}$ has no solutions for x.

Solving for a in terms of x gives $a = \frac{(x-4)(2x+7)}{2x-5} - x = \frac{4x-28}{2x-5}$. The first thing we observe is that $a = 2$ is not in the range of this expression. Secondly we note from the original equation that $x \neq -\frac{7}{2}$, and substituting this value into the expression for a gives $a \neq \frac{7}{2}$, and $(2)\left(\frac{7}{2}\right) = 7$.

6.) The points $(12,7)$, $(8 - 3\sqrt{3}, 4 + 4\sqrt{3})$, and $(4,1)$ are three vertices of a regular hexagon whose area is $a\sqrt{b}$ units, where a and b are integers with b having no perfect square factors larger than 1. Find the product ab.

See the diagram (not drawn to scale). By using the distance formula, we observe that the three vertices are equidistant from each other, and each vertex is 10 units apart. The only we three vertices of a regular hexagon can form an equilateral triangle is if all three sides of the triangle are diagonals (as shown). The area of an equilateral triangle with a side length of 10 is $25\sqrt{3}$, and since this is half the area of the hexagon, the area of the hexagon is $50\sqrt{3}$, so $ab = 150$.