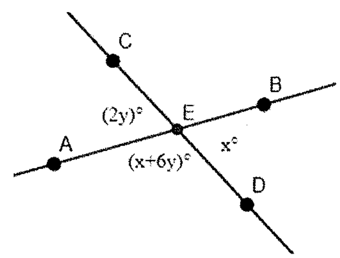
FAIRFIELD COUNTY MATH LEAGUE 2019-2020

Match 6 Round 1 Arithmetic: Lines & Angles

1.) 36°

2.) 16 minutes, 22 seconds

3.) 135° or 164°



1.) Consider the diagram to the right (not necessarily drawn to scale) of lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersecting at point E. Find $m\angle BED$ in degrees.

Note that $x = 2y$ since $\angle BED \cong \angle CEA$. Also $2y + x + 6y = 180$, so $2y + 2y + 6y = 180 \rightarrow y = 18$, so $x = 36$.

2.) After an analog clock strikes 12:00, what is the fewest number of minutes and seconds that must pass before the hands form a 90-degree angle? Round your answer to the nearest whole number of seconds.

Note that the hour hand on a clock moves one-half of a degree per minute and the minute hand moves 6 degrees per minute. Therefore, $\frac{1}{2}x + 90 = 6x$, or $x = \frac{180}{11} = 16\frac{4}{11}$ minutes, which to the nearest second is 16 minutes and 22 seconds.

3.) Consider kite $ABCD$ with $AB = AD$ and $BC = CD$. If $m\angle C = (3x + 150)^{\circ}$, $m\angle A = (x^2)^{\circ}$, and $m\angle B = 4m\angle A$, find all possible values in degrees of $m\angle C$.

All four angles must have measures that sum to 360 degrees, and by the information given we know $m\angle B = m\angle D = (4x^2)^{\circ}$, so $x^2 + 4x^2 + 4x^2 +$

$3x + 150 = 360 \rightarrow 9x^2 + 3x - 210 = 0 \rightarrow 3x^2 + x - 70 = 0 \rightarrow (x + 5)(3x - 14) = 0 \rightarrow x \in \{-5, \frac{14}{3}\}$. Neither value for x is extraneous as both produce four positive angle measures that sum to 360, so $m\angle C \in \{135^{\circ}, 164^{\circ}\}$.

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Match 6 Round 2 Algebra 1: Literal Equations

1.) $n = \frac{FV}{ART}$

2.) $c = \frac{ad}{x} - bd$ or $c = \frac{ad-bdx}{x}$

3.) $x = \frac{3}{y} + 2$ or $x = \frac{2y+3}{y}$

1.) If $PV = nRT$ and $F = PA$, find n in terms of $V, F, A, R,$ and T.

Substituting $P = \frac{F}{A}$, we get $\frac{FV}{A} = nRT$, so $n = \frac{FV}{ART}$.

2.) If $x = \frac{a}{b+\frac{c}{d}}$, find c in terms of $a, b, d,$ and x.

Multiplying both sides by $b + \frac{c}{d}$ and dividing by x yields $b + \frac{c}{d} = \frac{a}{x}$.
Subtracting b and multiplying by d gives $c = \frac{ad}{x} - bd$.

3.) If $x \neq 2$ and $y \neq 0$, find x in terms of y given $x^3y + 4xy - 12 = 2x^2y + 3x^2 + 8y$.

Grouping all terms with y on one side and all terms without y on the other side and factoring by grouping yields $y(x^3 - 2x^2 + 4x - 8) = 3x^2 + 12 \rightarrow y(x^2 + 4)(x - 2) = 3(x^2 + 4) \rightarrow y(x - 2) = 3 \rightarrow x = \frac{3}{y} + 2$.

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Match 6 Round 3 Geometry: Solids & Volume

1.) $\pi\sqrt[3]{36}$

2.) 8

3.) $\frac{1}{4}$

1.) What is the surface area of a sphere with a volume of π?

Starting with $\pi = \frac{4}{3}\pi r^3 \rightarrow r^3 = \frac{3}{4} \rightarrow r = \frac{\sqrt[3]{6}}{2}$, so $4\pi r^2 = 4\pi(\frac{\sqrt[3]{36}}{4}) = \pi\sqrt[3]{36}$.

2.) The center of a given cube is one unit away from any vertex. Find the surface area of the cube.

The distance from the center of the cube to one vertex is half the main diagonal of the cube. For a cube of side length s, the Pythagorean theorem can be used to find the length of the diagonal of one face is $s\sqrt{2}$ and the length of the main diagonal is $s\sqrt{3}$. Setting $s\sqrt{3} = 2$ gives $s = \frac{2}{\sqrt{3}}$, and $6s^2 = 6(\frac{4}{3}) = 8$.

3.) A solid cylinder of radius r and height h is sliced in half parallel to either base, and each resulting smaller solid cylinder has $\frac{3}{5}$ the surface area of the original cylinder. Find $\frac{r}{h}$.

Setting the surface area of a smaller cylinder equal to $\frac{3}{5}$ the surface area of the original cylinder gives $2\pi r^2 + 2\pi r(\frac{h}{2}) = \frac{3}{5}(2\pi r^2 + 2\pi rh) \rightarrow 2\pi r^2 + \pi rh = \frac{6}{5}\pi r^2 + \frac{6}{5}\pi rh \rightarrow \frac{4}{5}\pi r^2 = \frac{1}{5}\pi rh \rightarrow \frac{4}{5}r = \frac{1}{5}h \rightarrow \frac{r}{h} = \frac{1}{4}$.

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Match 6 Round 4 Algebra 2: Radical Expressions & Equations

1.) $\frac{\sqrt{6}}{3}$

2.) $x = \frac{4}{3}$

3.) $x = -\frac{15}{32}$ or $x = 40$

1.) Express in simplest radical form: $\frac{8^{5/6} \cdot 27^{1/6}}{12}$

We can rewrite the fraction as $\frac{2^{5/2} \cdot 3^{1/2}}{2^2 \cdot 3}$, which simplifies to $\frac{2^{1/2}}{3^{1/2}} = \frac{\sqrt{6}}{3}$.

2.) Solve for all real values of x: $\sqrt{x} + \sqrt{3} = \sqrt{x+7}$

Squaring both sides of the equation gives $x + 3 + 2\sqrt{3x} = x + 7$, yielding $2\sqrt{3x} = 4 \rightarrow 3x = 4 \rightarrow x = \frac{4}{3}$, which is not an extraneous solution.

3.) Solve for all real values of x: $4(2x+1)^{3/4} - \sqrt[4]{2x+1} = 12\sqrt{2x+1} - 3$

Letting $u = (2x+1)^{1/4}$, the equation becomes $4u^3 - u = 12u^2 - 3$, which set equal to zero becomes $4u^3 - 12u^2 - u + 3 = 0$. Terms can be grouped and factored to make $(4u^2 - 1)(u - 3) = 0$, so $u \in \{\frac{1}{2}, -\frac{1}{2}, 3\}$. Note that $-\frac{1}{2}$ is extraneous since u must be a positive number. Setting $\sqrt[4]{2x+1} = \frac{1}{2}$ and $\sqrt[4]{2x+1} = 3$ gives the answers of $x = -\frac{15}{32}$ or $x = 40$.

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Match 6 Round 5 Precalculus: Polynomials & Advanced Factoring

1.) $(x+1)(x+6)(2x-5)$

2.) 1 or 17

3.) $(-18,12)$

1.) Factor into three binomials with integer coefficients: $2x^3 + 9x^2 - 23x - 30$.

Observe that allowing x to be -1 makes the expression zero. Dividing out a factor of $x+1$ gives $(x+1)(2x^2+7x-30)$. The quadratic factor can be factored further through splitting the middle term and grouping, giving $(x+1)(x+6)(2x-5)$.

2.) The remainder when $f(x) = x^2 + 2kx + 3k^2$ is divided by $x+2$ is 8. Find all possible values of $f(1)$.

By the remainder theorem, we know $f(-2) = 8$, so $3k^2 - 4k + 4 = 8$, so $3k^2 - 4k - 4 = 0$, so $k = 2$ or $k = -\frac{2}{3}$. This gives two possible equations for $f(x)$; if $f(x) = x^2 + 4x + 12$, then $f(1) = 17$, but if $f(x) = x^2 - \frac{4}{3}x + \frac{4}{3}$, then $f(1) = 1$.

3.) The polynomial $f(x) = x^4 + ax^2 + bx + 80$ with real coefficients a and b has a zero of $x = 3 - i$. Find the ordered pair (a, b).

We can construct a quadratic factor of $f(x)$ knowing that two zeros are $3 \pm i$: $x^2 - (3+i+3-i)x + (3+i)(3-i) \rightarrow x^2 - 6x + 10$. From the structure of the original polynomial, we know that the sum of all four zeros must be zero and the product of all four zeros must be 80. Using the zeros we have, we know that our missing zeros must have a sum of -6 and a product of 8. Therefore the missing zeros must be -2 and -4. This gives $f(x) = (x+2)(x+4)(x^2 - 6x + 10) = x^4 - 18x^2 + 12x + 80$, so our ordered pair is $(-18,12)$.

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Match 6 Round 6 Miscellaneous: Counting & Probability

1.) 90720

2.) $\frac{41}{44}$

3.) $\frac{24}{49}$

1.) How many unique arrangements of the letters of "fairfield" are there?

There would be 9! permutations of the letters, but f and i are repeated, so we must divide out the repetitions, yielding $\frac{9!}{2!2!} = 90720$.

2.) A group of three students is to be chosen at random from a club consisting of 3 seniors, 4 juniors, and 5 sophomores. Find the probability that at least two students in the group of three will be from different grades.

The event specified is the complement of selecting a group of three students from the same grade. This has a probability of $\frac{\binom{3}{3}+\binom{4}{3}+\binom{5}{3}}{\binom{12}{3}} = \frac{1+4+10}{220} = \frac{3}{44}$. Therefore the probability of the specified event is $\frac{41}{44}$.

3.) A math league coach wants to incentivize her students to perform well in Match 6. She allows her two highest scorers to select one of two identical un labeled boxes at random and reach in and select a prize at random from that box. Box A contains 6 Panera gift cards and 4 Google Play gift cards. Box B contains 4 Panera gift cards and 5 Google Play gift cards. The first student selects a box at random and pulls out a Panera gift card. The second student selects the same box and pulls out a Google Play gift card. What is the probability the two students selected box A?

This can be phrased using conditional notation as P(Box A | First P, then G). Considering all possibilities, we have that both may have selected A with probability $\left(\frac{6}{10}\right)\left(\frac{4}{9}\right)$ or both may have selected B with probability $\left(\frac{4}{9}\right)\left(\frac{5}{8}\right)$, Therefore the probability that the two students selected Box A is $\frac{\left(\frac{6}{10}\right)\left(\frac{4}{9}\right)}{\left(\frac{6}{10}\right)\left(\frac{4}{9}\right)+\left(\frac{4}{9}\right)\left(\frac{5}{8}\right)} = \frac{24}{49}$.

Team Round .

FAIRFIELD COUNTY MATH LEAGUE 2019-2020 Match 6 Team Round

1.) $x = \frac{y+z}{2}$

2.) $(2,4,-50,-61)$

3.) $b = y + 2$

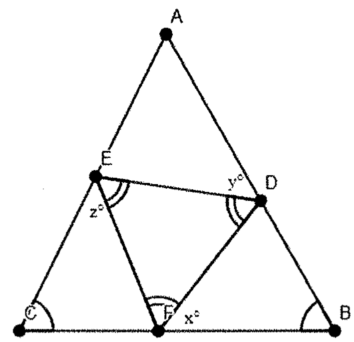
4.) $\frac{3\pi}{16}$

5.) $a = 2$ or $a < \frac{3}{2}$

6.) $y = \frac{z+\sqrt{z^2+4}}{2}$

1.) Consider ΔABC with $\angle B \cong \angle C$, point D on $\overline{AB}$, point E on $\overline{AC}$, and point F on $\overline{BC}$. If $m\angle DFB = x^o$, $m\angle ADE = y^o$, and $m\angle FEC = z^o$, and ΔDEF is equilateral, find x in terms of y and z.

Consider the diagram to the right. We know $m\angle FDB = (120 - y)^o$, and so $m\angle B = m\angle C = (180 - (x + 120 - y))^o = (60 - x + y)^o$. Using this along with $m\angle EFC = (120 - x)^o$, we can use triangle CEF to set up $60 - x + y + z + 120 - x = 180$, which gives $x = \frac{y+z}{2}$.



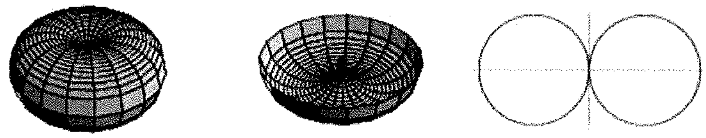
2.) The polynomial $f(x) = 4x^3 - 24x^2 - 2x + 7$ can be written in the form $f(x) = a(x-p)^3 + b(x-p) + c$ for real constants a, b, c, and p. Find the ordered quadruple (p, a, b, c).

Noticing that $f(x) = 4(x^3 - 6x^2) - 2x + 7$ and $(x-2)^3 = x^3 - 6x^2 + 12x - 8$, we can set up $f(x) = 4(x^3 - 6x^2 + 12x - 8) + b(x-2) + c$. This yields the relationships $4(12x) + bx = -2x$ and $4(-8) - 2b + c = 7$. From the first relationship we know $b = -50$, and using that in the second gives $c = -61$.

3.) A bag contains 3 red marbles, b blue marbles, and y yellow marbles. It is known that the probability of randomly drawing a blue marble followed by a yellow marble without replacement is p. If an additional blue marble is added to the original bag before any marbles are drawn, the probability of randomly drawing a blue marble followed by a yellow marble without replacement is still p. Find b in terms of y.

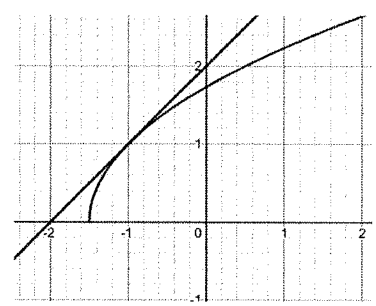
The probability of drawing a blue marble followed by yellow in the initial setup is $\left(\frac{b}{b+y+3}\right)\left(\frac{y}{b-1+y+3}\right)$. The probability of drawing a blue followed by a yellow in the second setup is $\left(\frac{b+1}{b+1+y+3}\right)\left(\frac{y}{b+y+3}\right)$. Setting the probabilities equal we get $\left(\frac{b}{b+y+3}\right)\left(\frac{y}{b+y+2}\right) = \left(\frac{b+1}{b+y+4}\right)\left(\frac{y}{b+y+3}\right)$. Note we can simplify immediately by dividing by sides by y and multiplying both sides by $b + y + 3$ to get $\frac{b}{b+y+2} = \frac{b+1}{b+y+4}$. Cross multiplication yields $b(b + y + 4) = (b + y + 2)(b + 1) \rightarrow b^2 + by + 4b = b^2 + by + 3b + y + 2$, which simplifies to yield $b = y + 2$.

4.) Consider a torus where a plane slicing vertically through the center shows tangent circular cross-sections, as shown to the right. If one of these circles has a diameter of R, consider also a sphere with a great circle cross-section having a radius of R. If V_T is the volume of the torus and V_S is the volume of the sphere, find the exact value of $\frac{V_T}{V_S}$.



The area of one circular cross-section of the torus is $\pi\left(\frac{R}{2}\right)^2$, and the volume of the torus is this area multiplied by the circumference of the circle formed by the centers of the circular cross-sections, which is $2\pi\left(\frac{R}{2}\right)$, making the volume $\frac{\pi^2 R^3}{4}$. Dividing this quantity by the volume of the sphere, $\frac{4}{3}\pi R^3$, gives $\frac{3\pi}{16}$.

5.) Find all real numbers a such that the equation $x + a = \sqrt{2x + 3}$ has exactly one real solution for x.



There will be one solution for x whenever the graph of $y(x) = x + a$ intersects the graph of $y(x) = \sqrt{2x + 3}$ exactly once. This will occur when the linear graph is tangent to the radical graph (see first figure) and when the linear graph lies below the radical graph so that it only intersects once. For the first case, squaring both sides of the equation gives $x^2 + 2ax + a^2 = 2x + 3$. The discriminant of this quadratic equation in terms of x is $(2a - 2)^2 - 4(a^2 - 3)$, and setting this equal to 0 gives $a = 2$. For the second case, we can note that the least value for a for which the linear graph will intersect the radical graph twice is $\frac{3}{2}$ since the x $-$intercept of the line will be $(-\frac{3}{2}, 0)$, which is the vertex of the radical graph. Therefore, the set of all values a is $a = 2$ or $a < \frac{3}{2}$.

6.) If $x < 0, y > 0$, and $z > 0$, solve for y in terms of z only given $z = x + 2y$ and $z^2 = x^2 + xy + y^2 - 3$.

Squaring the first equation and subtracting the second equation gives $0 = 3xy + 3y^2 + 3$. Solving for x in terms of y gives $x = -\frac{1}{y} - y$. Substituting back into the first equation gives $z = -\frac{1}{y} - y + 2y \rightarrow z = -\frac{1}{y} + y \rightarrow y^2 - zy - 1 = 0$. This gives $y = \frac{z \pm \sqrt{z^2+4}}{2}$, but since $y > 0$, the only correct solution is $y = \frac{z+\sqrt{z^2+4}}{2}$.