MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2009 - SOLUTIONS

Round 1 Arithmetic and Number Theory

1. Since $3^5 = 243$, base 3 is too small. Since $5^5 = 3125$, base 5 is too large. Thus, the base is $\boxed{4}$. Here, $133120_4 = 2009_{10}$.

2. Since $N = p(p^2 + 1)$ and N has 4 factors, those factors must be $1, p, p^2 + 1$, and $p(p^2 + 1)$. This means that $p^2 + 1$ must be prime so p must be even. Hence, $p = 2$ making $\boxed{N = 10}$.

3. There is 1 way to use ten 1 by 1 squares. There are ${}_8C_1 = 8$ ways to use seven 1 by 1's and one 1 by 3. There are ${}_6C_2 = 15$ ways to use four 1 by 1's and two 1 by 3's and there are ${}_4C_3 = 4$ ways to use one 1 by 1 and three 1 by 3's. This makes a total of $\boxed{28}$ different arrangements.

Round 2 Algebra 1

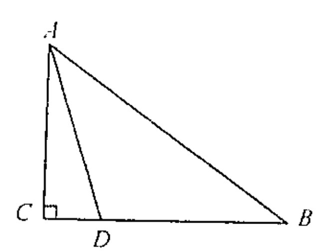
1. From $x^2 - xy + y^2 = \dfrac{42}{x + y}$ and $x^2 + xy + y^2 = \dfrac{12}{x - y}$, we obtain $x^3 + y^3 = 42$ and $x^3 - y^3 = 12$. Adding gives $2x^3 = 54$ so $\boxed{x = 3}$.

2. $|x - p| < d \to -d < x - p < d \to p - d < x < p + d$. For there to be 7 integral solutions, $(p + d) - (p - d) = 8$ since that gives p plus 3 integers on either side of p. Thus, $2d = 8$ gives $\boxed{d = 4}$.
   Alternate solution: The value of p is irrelevant so let $p = 0$. The solutions are the integers -3 to 3.

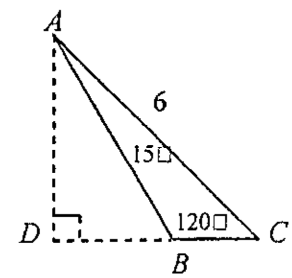
3. Since the median of the new set will be 10, the average must also be 10 so we have $\dfrac{7 + 8 + 10 + x + y}{5} = 10 \to x + y = 25$. The ordered pairs are $\boxed{(11,14),(12,13)}$.

Round 3 – Geometry

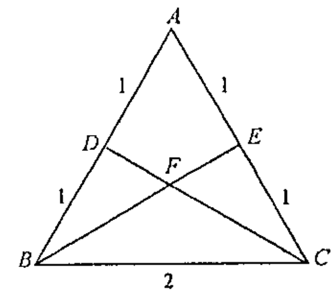
1. Let $CD = x$, then $AD = x\sqrt{10}$, giving
   $\left(x\sqrt{10}\right)^2 - x^2 = AC^2 \to AC = 3x$. Thus, $\dfrac{AC}{CD} = 3$.



2. Drop the altitude from A to $\overline{BC}$. Since $m\angle C = 45°$, ΔADC is a 45-45-90 triangle so $AD = DC = \frac{6}{\sqrt{2}} = 3\sqrt{2}$. Since ΔADB is a 30-60-90 triangle, $DB = \frac{3\sqrt{2}}{\sqrt{3}} = \sqrt{6}$, making $BC = 3\sqrt{2} - \sqrt{6}$. The area of ΔABC equals $\frac{1}{2} \cdot 3\sqrt{2} \cdot \left(3\sqrt{2} - \sqrt{6}\right) = \boxed{9 - 3\sqrt{3}}$.



3. Without lost of generality, let the side of the triangle have a length of 2. Since the median is the altitude, $DC = EB = \sqrt{3}$. Since F is a trisection point, $BF = CF = \frac{2\sqrt{3}}{3}$ and $DF = EF = \frac{\sqrt{3}}{3}$. The perimeter of $ADFE = 2 + \frac{2\sqrt{3}}{3}$ and the perimeter of $BFC = 2 + \frac{4\sqrt{3}}{3}$.



The ratio of the perimeter of $ADFE$ to the perimeter of $BFC = \frac{6 + 2\sqrt{3}}{6 + 4\sqrt{3}} = \boxed{\sqrt{3} - 1}$.

Round 4 – Algebra 2

1. The other roots are $-2i$ and $1 + i$ so the polynomial equals $(x^2 + 4)(x^2 - 2x + 2) = x^4 - 2x^3 + 6x^2 - 8x + 8$. The sum of the coefficients is $\boxed{5}$.
Alternate Solution: In the product $(x^2 + 4)(x^2 - 2x + 2)$, assigning $x = 1$ will give the sum of the coefficients, namely 5.

2. $f(x)$ could be any of the following ten expressions:
$3x^3$, $2x^3 + x^2$, $2x^3 + x$, $2x^3 + 1$, $x^3 + 2x^2$, $x^3 + 2x$, $x^3 + 2$, $x^3 + x^2 + x$, $x^3 + x^2 + 1$, and $x^3 + x + 1$. The ten corresponding values of $f(2)$ are
24, 20, 18, 17, 16, 12, 10, 14, 13, 11. The largest is 24, the smallest is 10, the sum is $\boxed{34}$.

3. $a_n = 2 + (n - 1)2$ so $S_n = \frac{(2 + 2 + (n - 1)2)n}{2} = (n + 1)n$. The product is divisible by 7 if n is 1 less than a multiple of 7 or is a multiple of 7. Thus 6, 7, 13, 14, . . . , 2001, 2002, and 2008, 2009 are the values of n that work. Counting the number of terms in 7, 14, . . . , 2009, we have $\frac{2009 - 7}{7} + 1 = 287$ terms. The answer is twice that or $\boxed{574}$.

Round 5 – Analytic Geometry

1. $f(1) = 5 + k$ and $f(2) = 18 + k$. We get a change in sign for $k = -6$ to -17 so there are $\boxed{12}$ values of k that yield a zero.

2 Horizontal asymptote: $y = \frac{1}{2}$. Thus, $\frac{x^3 + 5x}{2x^3 - x^2 + k} = \frac{1}{2} \rightarrow x^2 + 10x - k = 0$. To obtain two solutions the discriminant must be positive, so $10^2 + 4k > 0 \rightarrow \boxed{k > -25}$.

3. Using determinants we have $\frac{1}{2}\left(\begin{vmatrix}1 & 2\\ k & 8\end{vmatrix} + \begin{vmatrix}k & 8\\ 4 & k\end{vmatrix} + \begin{vmatrix}4 & k\\ 1 & 2\end{vmatrix}\right) = \pm 9$. The reason for the ± is that there are two possible orders for the points and so we must allow for clockwise and counterclockwise orderings. This simplifies to $k^2 - 3k - 16 = \pm 18$, yielding either $k^2 - 3k - 34 = 0$ or $k^2 - 3k + 2 = 0$. In each case the sum of the solutions for k is 3 so the answer is $\boxed{6}$. Note: in the first case $k = \frac{3 \pm \sqrt{145}}{2}$ and in the second case $k = 1$ or 2.

Round 6 – Trig and Complex Numbers

1. Since $\frac{1}{2}(6)(8)\sin B = 20$, $\sin B = \frac{5}{6}$. Thus $\cos B = \sqrt{1 - \frac{25}{36}} = \boxed{\frac{\sqrt{11}}{6}}$.

Alternate Solution: The altitude from A must equal 5 since 8(5)/2 = 20. $\sin A = \frac{5}{6}$ and $\cos A = \frac{\sqrt{11}}{6}$

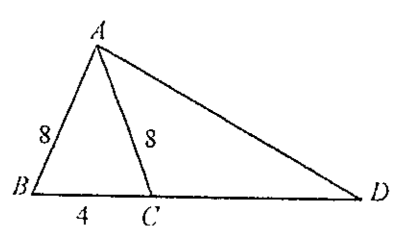
2. $\left|\frac{1}{1+i}\cdot\frac{1-i}{1-i} - a\right| = \left|\frac{1-i}{2} - a\right| = \left|\frac{1-2a}{2} - \frac{1}{2}i\right| = 1 \rightarrow \left(\frac{1-2a}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1 \rightarrow$

$4a^2 - 4a + 2 = 4 \rightarrow 2a^2 - 2a - 1 = 0 \rightarrow a = \frac{1 \pm \sqrt{3}}{2}$. Choose $\boxed{a = \frac{1+\sqrt{3}}{2}}$.

Alternate Solution: a must be the set of points on the unit circle with center $\left(\frac{1}{2}, -\frac{1}{2}\right)$. Since only real values of a are wanted, set $y = 0$ in $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 1$

3. Let $CD = 3x$ and $AD = 4x$. Draw altitude from A to $\overline{BC}$. Then $\cos ACD = -\cos ACB = -\frac{1}{4}$.

$9x^2 + 64 + 12x = 16x^2 \rightarrow (7x + 16)(x - 4) = 0 \rightarrow x = 4$
$\rightarrow$ AD = 16.



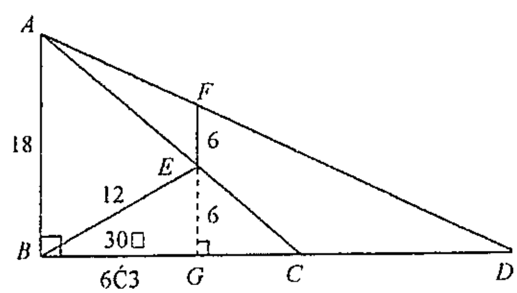
Team Round

1. Consider the one particular placing of the digits into the top row as shown on the right. We must place 3 and 4 into the bottom row of the left box and 1 and 2 in the bottom row of the right box. There are 2 ways to put each pair of numbers in the appropriate box. There are $4! = 24$ ways to put 1, 2, 3, and 4 in the top row and for any top row there are $2 \cdot 2 = 4$ ways to place the remaining numbers in the bottom row. Answer: $4! \cdot 4 = \boxed{96}$.

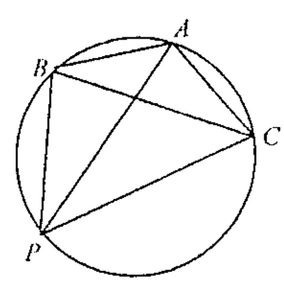
1	2	3	4

2. Let $x = \sqrt{n - \sqrt{n - \sqrt{n - \sqrt{n - \text{K}}}}}$, then $x^2 = n - \sqrt{n - \sqrt{n - \sqrt{n - \text{K}}}} = n - x$. Thus, $x^2 + x = n$. Consequently, n is an integer between 1 and 2009 which is the product of the integers x and $x + 1$. If $x = 1$, then $n = 2$, if $x = 44$, $n = 1980$ and if $x = 45$, $n = 2070$. Clearly, x can take on all integer values from 1 to 44 inclusive so the answer is $\boxed{44}$.

3. Extending $\overline{EF}$ to G we have the lengths as marked. Since $\Delta ABC : \Delta EGC$, we have $\dfrac{AB}{EG} = \dfrac{BC}{GC} \rightarrow$ $\dfrac{18}{6} = \dfrac{6\sqrt{3} + GC}{GC} \rightarrow GC = 3\sqrt{3}$. Since $\Delta ABD : \Delta FGD$, then $\dfrac{AB}{FG} = \dfrac{BD}{GD} \rightarrow$ $\dfrac{18}{12} = \dfrac{9\sqrt{3} + CD}{3\sqrt{3} + CD}$, making $\boxed{CD = 9\sqrt{3}}$.



4. Draw $\overline{PA}$ creating quadrilateral $ABPC$ with diagonals $\overline{BC}$ and $\overline{AP}$. By Ptolemy's Theorem, $AB \cdot PC + AC \cdot BP = BC \cdot PA$. Since $m\overset{a}{A}B = m\overset{a}{A}C = 60°$, then $AB = AC$ and both equal the radius r of the circle. By the Law of Cosines, $BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos 120°$. Using r



we have $BC^2 = 2r^2 - 2r^2\left(-\frac{1}{2}\right) \rightarrow BC = r\sqrt{3}$. Substituting $BA = AC$ into Ptolemy's Theorem gives $AB(PC + BP) = BC \cdot PA \rightarrow \frac{PC + PB}{PA} = \frac{BC}{AB} = \frac{r\sqrt{3}}{r}$. Thus, $\boxed{\frac{PC + PB}{PA} = \sqrt{3}}$.

5. Multiply the three equations together and after some simplifying we obtain $xyz + \frac{1}{xyz} + x + \frac{1}{x} + y + \frac{1}{y} + z = abc$. Since $n + \frac{1}{n} \geq 2$ for positive n, the left side is greater than or equal to 8. Thus, $abc \geq 8$. The minimum value is $\boxed{8}$.

6. Draw radius $\overline{AE}$, creating isosceles triangles ADE, BDE, and BCE. Let $m\angle A = \theta$, making $m\angle BDE = m\angle BED = 2\theta$. We also have $m\angle DBE = 180 - 4\theta$, $m\angle BEC = 180 - 3\theta$, making $m\angle EBC = 6\theta - 180$ and $m\angle ABC = 2\theta$. Since $\overline{BC}$ is the shortest side, then $m\angle A < m\angle BCA \rightarrow \theta < 180 - 3\theta$ $\rightarrow \theta < 45°$. Since $m\angle ABC > m\angle ABE$, then $2\theta > 180 - 4\theta$, making $\theta > 30°$. The other possible inequalities did not yield further restrictions so $\boxed{30 < \theta < 45}$.