NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2008

Round 1: Arithmetic and Number Theory

1. ____________________

2. ____________________

3. ____________________

1. Consider the natural numbers between the largest 2-digit prime number and the smallest 3-digit prime number. If each one is divided by their largest square factor, what is the product of the quotients?

2. My measure of an edge of a cube is within 1 cm of its true length of 100 cm. If the volume of the cube is calculated from my measurement, the result is accurate to within N cm^3 of the true volume. Determine N.

3. Let p be a prime. If $p + 2p + 3p + \cdots + 24p = N$ and N has exactly 36 positive factors, determine the least possible value for p.

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Round 2: Algebra 1

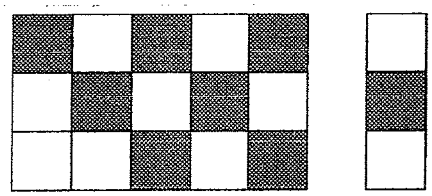
1. ______________________

2. ______________________

3. ______________________

1. Four consecutive whole numbers are such that the product of the first and fourth is 55 less than the square of the third. What is the sum of these four numbers?

2. How many of the 3 by 1 strips shown at the right of the 3 x 5 rectangle need to be attached to its end in order that the percentage of shaded squares in the new rectangle first drops below 40%?



3. Solve the system for x: $1003x + 1001y = 817$ and $1005x + 1007y = 1191$.

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Round 3: Geometry

1. ____________________

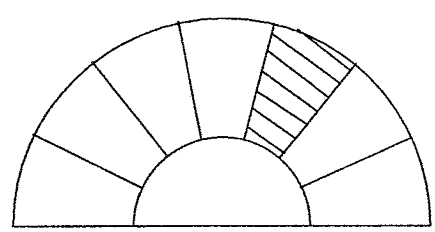
2. ____________________

3. ____________________

1. In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect in point E. If $AE = 8$, $EB = 10$, and CE is 1 more than 3 times ED, what is the length of $\overline{CD}$?

2. In right triangle ABC, $m\angle C = 90°$. D lies on $\overline{BC}$ such that $AD = DB = \frac{5}{3}CD$. Determine the numerical value of $\frac{AB}{CD}$.

3. The shaded region is congruent to the other six regions between the two semicircles. The area of the shaded region is 12 and the radius of the large semicircle is twice the radius of the small semicircle. Find the area of the large semicircle.



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Round 4: Algebra 2

1. ____________________

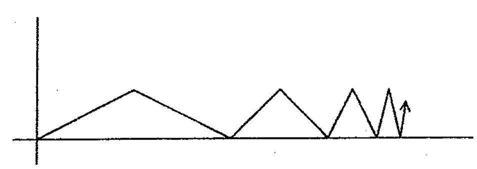
2. ____________________

3. ____________________

1. The function: $y = (\log_7 x)^2 - 4\log_7 x - 2$ is defined for positive real x. Find the sum of the coordinates of the lowest point on the graph.

2. When $x^5 + 200 - x^2(25x + 8)$ is factored completely over the reals, determine the sum of the absolute values of the constants of the factors.

3. The diagram shows an infinite number of isosceles triangles whose heights are the same. The area of the leftmost triangle is 9 and the slope of the left side of each triangle thereafter is twice the slope of the left side of the triangle immediately preceding it. Find the sum of the areas of all the triangles.



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Round 5: Analytic Geometry

1. ______________________

2. ______________________

3. ______________________

1. Determine all values of 'a' such that the lines with the following equations are perpendicular: $a^2x + y = c$ and $\frac{x}{2-a} - y = d$.

2. Let $y = f(x)$ be a linear function with slope greater than 1 that passes through the origin. If the distance between $(1, f(1))$ and $(f(1), 1)$ is $\sqrt{6}$, determine the slope of $y = f(x)$.

3. The vertex of the parabolic function $f(x) = ax^2 + bx + c$ is (2, –9). Find all possible values of 'a' such that the graph of the parabola has two non-negative x-intercepts.

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Round 6: Trig and Complex Numbers

1. ______________________

2. ______________________

3. ______________________

1. If $\tan A = -\frac{3}{4}$, $\frac{\pi}{2} < A < \pi$, $\cot B = \frac{12}{5}$, and $0 < B < \frac{\pi}{2}$ determine the value of $\sin(2A + B)$.

2. If $p^2 = 3 - i\sqrt{7}$ and $q^2 = 3 + i\sqrt{7}$, find the positive value of $p + q$.

3. Given z and w are complex numbers such that $z^3 = 2 + 2i$ and $w^5 = 1 - i$. Let the solutions in polar form to the equations be $z_i, i = 0,1,2$ and $w_j, j = 0,1,2,3,4$. For each solution of the above equations, the angles (amplitudes) are between 0 and 2π, and are increasing in counterclockwise order. If $\frac{z_1}{w_0} =$ $r\text{cis}\theta$, determine the ordered pair (r, θ) where $0 \le \theta < \frac{\pi}{2}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2008

Team Round

1. ____________________ **4.** ____________________

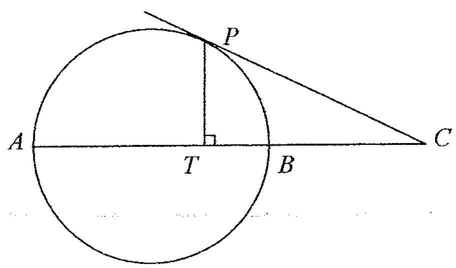
2. ____________________ **5.** ____________________

3. ____________________ **6.** ____________________

1. Point $A\left(1,\frac{1}{3}\right)$ lies on the graph of $y = \frac{x^2}{3}$. Point B lies on the graph in the first quadrant. If $\overline{AB}$ is a diameter of a circle that is tangent to the x-axis, find all points B.

2. Let the hour hand of a clock be represented by $\overline{OA}$ and the minute hand by $\overline{OB}$. It is now 3 pm. Determine the probability that at a randomly chosen time in the next hour $\angle BOA$ is obtuse.

3. $\overline{AC}$ contains diameter $\overline{AB}$, $\overline{CP}$ is tangent to the circle at P, and $\overline{PT} \perp \overline{AC}$. If $AT = 12$ and $BT = 8$, find CT.



4. Integers from 10,000 to 99,999 are considered to be <u>weakly increasing</u> if, reading from left to right, the first two digits and the last three digits each form a strictly increasing sequence. How many <u>weakly increasing</u> 5-digit numbers are there?

5. Let $f_1(x) = \frac{1}{x-1}$ and $f_n(x) = \left(1 + f_{n-1}(x)\right)^{-1}$ for $n \geq 2$. If $g_n(x) = f_1(x) \cdot f_2(x) \cdots f_n(x)$, determine the value of $g_5(2)$.

6. Let $\{a_n\}$ and $\{b_m\}$ be arithmetic sequences that start with $a_1 = 3,\ a_2 = 12$ and $b_1 = 4,\ b_2 = 15$ respectively. For $1 \leq n, m \leq 100$, find the number of ordered pairs (n, m) such that $\left|a_n - b_m\right| = 1$.

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Answer Sheet

Round 1

1. 22
2. 30,301
3. 7

Round 2

1. 210
2. 6
3. –92

Round 3

1. 21
2. $\frac{4\sqrt{5}}{3}$
3. 112

Round 4

1. 43
2. 16
3. 18

Round 5

1. 1, –2
2. $1+\sqrt{3}$
3. $a \geq \frac{9}{4}$

Round 6

1. $-\frac{253}{325}$
2. $\sqrt{14}$
3. $\left(\sqrt[5]{4}, \frac{2\pi}{5}\right)$

Team

1. $(3,3)$, $\left(\frac{3}{5}, \frac{3}{25}\right)$
2. $\frac{5}{11}$
3. 48
4. 4320
5. $\frac{1}{8}$
6. 19