FAIRFIELD COUNTY MATH LEAGUE 20~~17~~18-20~~18~~19

Match 6 Round 1
Geometry: Lines
and Angles

1.) ____________90__________degrees

2.) ___________142__________degrees

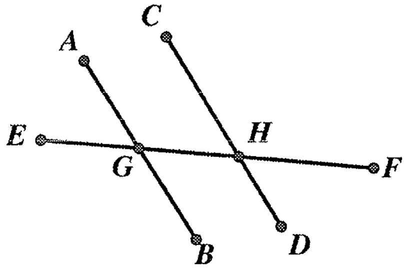
Note: Figures not necessarily
Drawn to scale

3.) ____________94__________degrees

1.)_ A triangle is formed by connecting points A(-2,1), B(1,3), and C(5,-3). What is the sum of the measures of $\angle BAC$ and $\angle BCA$?

$\overline{AB}$ has slope $\frac{3-1}{1-(-2)}=\frac{2}{3}$. $\overline{BC}$ has slope $\frac{-3-3}{5-1}=\frac{-6}{4}=\frac{-3}{2}$. So $\overline{AB}$ and $\overline{BC}$ are perpendicular to each other and form a 90 degree angle at B. The sum of the angles of a triangle is 180 degrees, so the other two angles must add to 180-90 = 90 degrees.

2.)_ $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$. The lines are cut by transversal $\overleftrightarrow{EF}$, which intersects line $\overleftrightarrow{AB}$ at G and $\overleftrightarrow{CD}$ at H. If $m(\angle HGB)=(31+3x)^\circ$ $m(\angle CHF)=(6x+29)^\circ$, find $m(\angle AGE)+m(\angle FHD)$.



$m(\angle HGB)=m(\angle AGE)$ by vertical angles. $m(\angle AGE)+m(\angle CHF)=180^\circ$ since they are same-side exterior angles.

$$(31+3x)+(6x+29)=180$$

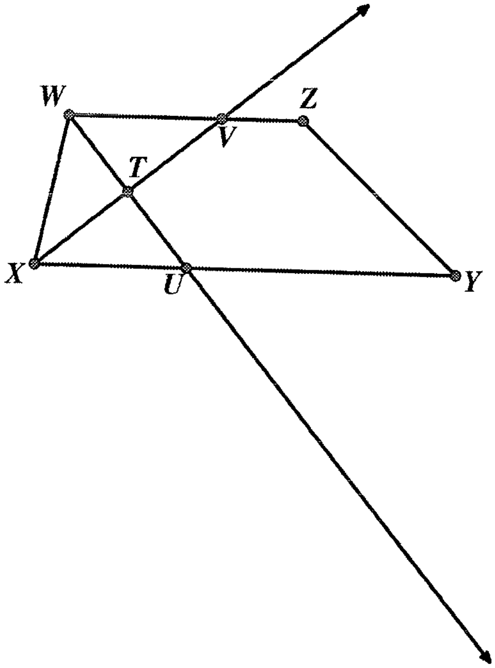
$$9x+60=180$$

$$9x=120$$

$$x=\frac{120}{9}=\frac{40}{3}$$

$\angle AGE$ and $\angle FHD$ are both congruent to $\angle HGB$, which has measure $31+3(\frac{40}{3})=71$ degrees. The sum is 2*71= 142 degrees.

3.) In trapezoid WXYZ, $\overline{WZ}$ is parallel to $\overline{XY}$, $\overline{XV}$ bisects $\angle WXY$ and $\overline{WU}$ bisects $\angle XWZ$. The bisectors meet at T. $\angle WXT = (6x-5)°$ and $\angle XWT = (x^2-17)°$. Find the measure of $\angle ZWX$



Same-side interior angles of a trapezoid for these sides must add to 180 degrees, so the sum of the bisected angles must add to 90 degrees.

$$(6x-5)+(x^2-17)=90$$

$$x^2+6x-22=90$$

$$x^2+6x-112=0$$

$$(x-8)(x+14)=0$$

$$x=8$$

since -14 makes (6x-5) negative.

The measure of $\angle ZWX$ is $2(x^2-17)=2(64-17)=2(47)=94$ degrees.

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Match 6 Round 2
Algebra: Literal
Equations

1.) ______ z= $3x+4y-5$____________

2.) ___R_3=_ $\dfrac{RR_1R_2}{R_1R_2 - RR_2 - RR_1}$ ______

3.) ______ y = __x+2_______________

1.) Solve for z in terms of x and y: $3x+5y-z=15+2z-6x-7y$

$$3x+5y-z=15+2z-6x-7y$$

$$-3z=-9x-12y+15$$

$$z=\frac{-9x-12y+15}{-3}=3x+4y-5$$

2.)_ If three resistors with resistance R_1, R_2, and R_3 are arranged in an electric circuit in parallel, the formula for the equivalent resistance R is found by the formula $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$. Solve this equation for R_3. Express your answer as a single fraction.

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$(\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3})RR_1R_2R_3$$

$$R_1R_2R_3 = RR_2R_3 + RR_1R_3 + RR_1R_2$$

$$R_1R_2R_3 - RR_2R_3 - RR_1R_3 = RR_1R_2 .$$

$$R_3(R_1R_2 - RR_2 - RR_1) = RR_1R_2$$

$$R_3 = \frac{RR_1R_2}{R_1R_2 - RR_2 - RR_1}$$

3.) If x≠2 , solve for y in terms of x:

$$x^4 - x^3y - 2yx + 8y + x^2y = 2yx - x^2y + 16$$

$$x^4 - x^3y - 2yx + 8y + x^2y = 2yx - x^2y + 16$$

$$x^4 - 16 = x^3y + 2yx - 8y - x^2y + 2yx - x^2y$$

$$x^4 - 16 = x^3y + 4yx - 8y - 2x^2y$$

$$x^4 - 16 = y(x^3 + 4x - 2x^2 - 8)$$

$$x^4 - 16 = y(x(x^2 + 4) - 2(x^2 + 4))$$

$$x^4 - 16 = y(x - 2)(x^2 + 4)$$

$$(x^2 + 4)(x + 2)(x - 2) = y(x - 2)(x^2 + 4)$$

$$y = \frac{(x^2 + 4)(x + 2)(x - 2)}{(x - 2)(x^2 + 4)} = x + 2$$

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Match 6 Round 3
Geometry:
Solids and
Volumes

1.)________ $\frac{72}{\pi}$ __________ in^3_

2.)______ $288\pi - 192\sqrt{3}$_ cm^3

3.) __k=__$2\sqrt[3]{4}$ _____________

1) The lateral area of a cylinder (the surface area not including the bases) with height 8 inches is 48 in^2. What is the volume of the cylinder?

$$A = 2\pi rh$$

$$48 = 2\pi r(8)$$

$$r = \frac{48}{16\pi} = \frac{3}{\pi}$$

Volume = πr^2 h, so $V = \pi(\frac{3}{\pi})^2 * 8 = \pi\frac{9}{\pi^2} * 8 = \frac{72}{\pi}$

2. A cube is inscribed in a sphere of radius 6 cm. What volume is inside the sphere but outside the cube?

The radius of the sphere must be half the main diagonal of the cube. If the radius of the sphere is r, then to find the side s of the cube,

$$6 = \frac{1}{2}\sqrt{s^2 + s^2 + s^2} = \sqrt{3s^2} = 12, 3s^2 = 144, s = \sqrt{\frac{144}{3}} = \sqrt{48} = 4\sqrt{3}.$$

The volume of the sphere is $\frac{4}{3}\pi(6)^3 = 288\pi$. The volume of the cube is

$(4\sqrt{3})^3 = 4^3 * 3\sqrt{3} = 192\sqrt{3}$. The desired volume is $288\pi - 192\sqrt{3}$ cm^3

3. A cone is formed by rotating the line $y = \frac{2}{5}x$ from x=0 to x=10 around the y-axis. The plane y=k cuts the cone so that half of its volume is above the plane y=k. Find the value of k.

The original cone has base radius 10 and height $\frac{2}{5}(10) = 4$, so its volume is $\frac{1}{3}\pi(4*10^2) = \frac{400\pi}{3}$. When the height y is k, then x solves the proportion $\frac{k}{x} = \frac{4}{10}$, so $x = \frac{5}{2}k$. We need k such that $\frac{1}{3}\pi(k(\frac{5}{2}k)^2 = \frac{200\pi}{3}$

$$\frac{1}{3}\pi(k(\frac{5}{2}k)^2) = \frac{200\pi}{3}$$

$$\frac{25}{4}k^3 = 200$$

$$k^3 = \frac{800}{25} = 32$$

$$k = \sqrt[3]{32} = 2\sqrt[3]{4}$$

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Match 6 Round 4
Radical Expressions and Equations

1.) $\frac{29\sqrt{5}}{5}$

2.) $\sqrt[6]{1125}$

3.) 4

1.)_ Express as a single reduced fraction in simplest radical form:

$$3\sqrt{45}+\frac{4}{\sqrt{5}}-\sqrt{80}$$

$$3\sqrt{45}+\frac{4}{\sqrt{5}}-\sqrt{80}=$$

$$3(3\sqrt{5})+\frac{4}{\sqrt{5}}-4\sqrt{5}=$$

$$9\sqrt{5}+\frac{4\sqrt{5}}{5}-4\sqrt{5}=$$

$$5\sqrt{5}+\frac{4\sqrt{5}}{5}=$$

$$\frac{25\sqrt{5}}{5}+\frac{4\sqrt{5}}{5}=\frac{29\sqrt{5}}{5}$$

2) Express as a single radical: $\sqrt[3]{3}\sqrt{5}$

$$=3^{\frac{1}{3}}5^{\frac{1}{2}}=3^{\frac{2}{6}}5^{\frac{3}{6}}=(3^2 5^3)^{\frac{1}{6}}=\sqrt[6]{1125}$$

3._Solve for all real values of x:

$$\sqrt{2x+1}-\sqrt{x-3}=\sqrt{5x-16}$$

$$\sqrt{2x+1}-\sqrt{x-3}=\sqrt{5x-16}$$

$$2x+1+x-3-2\sqrt{2x+5}\sqrt{x-3}=5x-16$$

$$3x-2-2\sqrt{2x+5}\sqrt{x-3}=5x-16$$

$$-2\sqrt{2x+1}\sqrt{x-3}=2x-14$$

$$\sqrt{2x+1}\sqrt{x-3}=7-x$$

$$(2x+1)(x-3)=49-14x+x^2$$

$$2x^2-5x-3=49-14x+x^2$$

$$x^2+9x-52=0$$

$$(x+13)(x-4)=0$$

$$x=-13, x=4$$

-13 is extraneous, so x=4.

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Match 6 Round 5
Polynomials and
Advanced
Factoring

1.) ________144_______________

2.) ________7, -65, 135________

3.) __ ___________7_____________

1.) What is the remainder when $x^5 - 4x^3 + x^2 - x + 3$ is divided by $x - 3$?

If they know the remainder theorem, it's just $3^5 - 4*3^3 + 3^2 - 3 + 3 =$ $243 - 108 + 9 = 144$. Otherwise they do the long division, being careful since there is no x^4 term.

2.) $x^3 + ax^2 + bx + 7$ factors into three binomials with integer coefficients. Find all possible values of ab.

$x^3 + ax^2 + bx + 7$ must be $(x+r)(x+s)(x+t)$ where rst=7 and r,s,t integers.
Possibilities are
$(x-1)(x-1)(x+7)$
$(x-1)(x-7)(x+1)$
$(x+1)(x+1)(x+7)$

$(x-1)(x+1)(x-7)=x^3 - 7x^2 - x + 7$ ab=7
$(x-1)(x+7)(x-1)=x^3 + 5x^2 -13x + 7$, ab = -65
$(x+1)(x+1)(x+7) = x^3 + 9x^2 +15x +7$, ab = 135

3) A quartic polynomial $x^4 + ax^3 + bx^2 + cx + d$ where a, b, c, and d are integers has 1+i and 3-2i as two of its zeros. Find a+b+c+d.

Since coefficients are integers, if 1+i is a zero, so is 1-i. If 3-2i is a zero, so is 3+2i. The quadratic with zeros 1+i and 1-i has $\frac{-b}{a} = 2, \frac{c}{a} = 2$, so $x^2 - 2x + 2$, since the leading coefficient must be 1 to give x^4 (-1 will just require the second polynomial to also have leading coefficient -1, and the negatives cancel out).

The quadratic with zero 3-2i and 3+2i has $\frac{-b}{a} = 6, \frac{c}{a} = 13$, so $x^2 - 6x + 13$

Muliply $(x^2 - 2x + 2)(x^2 - 6x + 13) =$

$x^4 - 8x^3 + 27x^2 - 38x + 26$

-8 + 27 – 38 + 26 = -46 + 53 = 7

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Match 6 Round 6
Counting and
Probability

1.) ________4536________

2.) ________ $\frac{5}{12}$ ________

3.) _____(1,0), (3,1), (8,3)____

1.) How many 4 digit numbers N 1000≤N≤9999 have no repeating digits? (e.g., 4576 has no repeating digits, but 4546 has a repeating 4).

There are 9000 numbers overall. The number that have no repeating digits is 9*9*8*7 (since the first digit can't be a zero) = 4536.

2.)_ The names of five boys and five girls are placed into a hat and three names are drawn out without replacement. What is the probability that the names of exactly two girls were drawn?

The total number of possibilities is $_{10}C_3 = \frac{10*9*8}{3*2*1} = 10*3*4 = 120$. If 2 girls out of 5 and one boy out of 5 were drawn the numerator is $({}_5C_2)({}_5C_1) = (\frac{5*4}{2*1})(5) = 10*5 = 50$. The probability is $\frac{50}{120} = \frac{5}{12}$

3.) $_NP_R$ is the number of permutations of N objects taken R at a time. $_N C_R$ is the number of combinations of N objects taken R at a time. If 1≤N≤10, find all ordered pairs (N,R) such that ${}_N P_R = {}_N C_{R+1}$

We need $\frac{N!}{(N-R)!} = \frac{N!}{(N-(R+1))!(R+1)!}$

$(N-R)! = (N-R)(N-(R+1))!$ so this becomes

$\frac{N!}{(N-R)(N-(R+1))!} = \frac{N!}{(N-(R+1))!(R+1)!}$, after cancelling,

$\frac{1}{N-R} = \frac{1}{(R+1)!}$, so $N-R=(R+1)!$

Since 1≤N≤10, $(R+1)!$≤10, so that leaves R=0, R=1, R=2.

If R=0, N=1

If R=1, N=3

If R=2, N=8

so (1,0), (3,1), (8,2)

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Match 6
Team
Round

1.) _E =_______9A-1470_____ 4.) $(a-2)(a-4)(a^2+5)$

2.) ________3, 66, 731______ 5.) _____ $\frac{64}{81}$ _________

3.) ______ $8A\sqrt{3A}$ ________6.) M = 12, N = 20_____

1.)_ Angles A, B, C, D, and E are the five interior angles of a convex pentagon. The measure of $\angle B$ is equal to ten times the measure of the supplement of $\angle A$. Two times the measure of $\angle B$ is eighty degrees more than the measure of $\angle C$. The measure of $\angle D$ is thirty degrees more than the supplement of the measure of $\angle C$. Find the measure of $\angle E$ in terms of measure of $\angle A$. Use E for the measure of $\angle E$ and A for the measure of $\angle A$.

Let A,B,C,D,E stand for the measures of angles A,B,C,D,E

A+B+C + D+E=540
B=10(180-A)=1800-10A
2B=C+80, so C=2B-80=2(1800-10A)-80=3520-20A
D=30+(180-C), so D=210-(3520-20A)=20A-3310
A+(1800-10A)+(3520-20A)+(20A-3310)+E=540
-9A+2010+E=540
E=9A+540-2010=9A-1470.

2.) Find all values of x such that. $\sqrt{x-2}-6\sqrt[3]{x-2}+11\sqrt[6]{x-2}=6$

$$((x-2)^{\frac{1}{6}})^3-6((x-2)^{\frac{1}{6}})^2+11((x-2)^{\frac{1}{6}})-6=0$$

Let y=$(x-2)^{\frac{1}{6}}$. Then $y^3-6y^2+11y-6=0$. y=1 is clearly a solution since $1^3-6*1^2+11*1-6=0$, so y-1 is a factor. Divide $y^3-6y^2+11y-6$ by y-1 to get $y^2 - 5y + 6 = (y-3)(y-2)$, so the complete equation is

$$(y-1)(y-2)(y-3) = 0$$

$$y = 1,2,3$$

If $(x-2)^{\frac{1}{6}} = 1$, (x-2)=1, x=3

If $(x-2)^{\frac{1}{6}}$=2, x-2=64, x=66

If $(x-2)^{\frac{1}{6}}$=3, x-2=729, x=731.

Solution is 3, 66, 731.

3). A tetrahedron has surface area $24A\sqrt{3}$. Give the volume of the tetrahedron in terms of A in simplest radical form.

Each triangle has area $6A\sqrt{3}$. Since they are all equilateral triangles, to find the side of the triangle, set $\dfrac{s^2\sqrt{3}}{4} = 6A\sqrt{3}$

$$s^2 = 24A, s = 2\sqrt{6A}$$

Align the tetrahedron with one triangle as its base. The apex of the pyramid will be directly above the point that is two-thirds of the way from any vertex of the triangle along its altitude. The height of the tetrahedron is one of the legs of a right triangle, where the hypotenuse is the side of the tetrahedron and the other legs is one-third of the altitude of the triangle. To find the altitude of the triangle, break up the triangle into two 30-60-90's, so that the short side is $\sqrt{6A}$ and the long side is $\sqrt{3}\sqrt{6A} = 3\sqrt{2A}$ and take two thirds of that as the leg used to find the height of the tetrahedron.

$$(2\sqrt{2A})^2 + h^2 = (2\sqrt{6A})^2$$

$$8A + h^2 = 24A$$

$$h^2 = 16A, h = 4\sqrt{A}$$

The volume of the pyramid is one-third times the height times the area of the base, so

$$V = \frac{1}{3}(6A\sqrt{3})(4\sqrt{A})$$
$$= 8A\sqrt{3A}$$

4.) Factor into three binomials: $a^4 - 6a^3 + 13a^2 - 30a + 40$

$$\begin{aligned}
&a^4 - 6a^3 + 13a^2 - 30a + 40 \\
&= a^4 - 6a^3 + 8a^2 + 5a^2 - 30a + 40 \\
&= (a^4 - 6a^3 + 8a^2) + (5a^2 - 30a + 40) \\
&= a^2(a^2 - 6a + 8) + 5(a^2 - 6a + 8) \\
&= (a^2 - 6a + 8)(a^2 + 5) \\
&= (a-2)(a-4)(a^2+5)
\end{aligned}$$

5) Naphesa Collier of the UConn women's basketball team makes $\frac{2}{3}$ of her shots. If the shots are independent and she takes 5 shots in a game, what is the probability that she makes at least 3 of the 5 shots?

$$({}_5C_3)(\frac{2}{3})^3(\frac{1}{3})^2 + ({}_5C_4)(\frac{2}{3})^4(\frac{1}{3})^1 + ({}_5C_5)(\frac{2}{3})^5(\frac{1}{3})^0 =$$

$$\frac{10*8}{243} + \frac{5*16}{243} + \frac{1*32}{243} = \frac{80}{243} + \frac{80}{243} + \frac{32}{243} = \frac{192}{243} = \frac{64}{81}$$

6) A semi-regular polyhedron is composed of M pentagons and N hexagons. An edge of the polyhedron is formed when one side of one of the polygons meets a side of another polygon. A vertex of the polyhedron is formed when one corner of a pentagon meets corners of two hexagons. The polyhedron has 90 edges and 60 vertices. Find M and N.

By Euler's formula, F+V=E+2, so F+60=90+2, so F=32, then M+N = 32. Either use the edges or the vertices to find the second equation. You have 5M+6N sides of polygons altogether, divide by 2 to get the number of edges, so

$$\frac{5M+6N}{3}=60$$

$M+N=32$

$5M+6N=180$

$-5M-5N=-160$ $\frac{5M+6N}{2}=90$. If you used vertices, you could use

$5M+6N+180$

$N=20$

$M=32-20=12$

$\frac{5M+6N}{3}=60$. Either way, $5M+6N=180$. Solve the system

$M+N=32$ $-5M-5N=-160$ $N=20$

$5M+6N=180$, $5M+6N+180$, $M=32-20=12$