FAIRFIELD COUNTY MATH LEAGUE 2018-2019

Match 4 Round 1
Arithmetic:
Basic Statistics

1.) ________9________

2.) ________5____________

3.) ________$29____________

1.) What is the positive difference between the arithmetic mean and the median of the set of numbers of the form 2^N where N is an integer and $1 \le N \le 6$?

Arithmetic mean: $\frac{2+4+8+16+32+64}{6}=\frac{126}{6}=21$

Median: $\frac{8+16}{2}=12$

Difference is 21-12 = 9.

2.) The geometric mean of a set of numbers $\{a_1, a_2 ..., a_N\}$ is $\sqrt[N]{a_1 a_2 ... a_N}$. The geometric mean of 6 numbers is 45. Five of the numbers are 9, 27, 125, 225 and 243. What is the sixth number?

9, 27, 125, 225, and 243 are $3^2, 3^3, 5^3, 3^2 * 5^2, 3^5$. 45 = $3^2 * 5$,

If N is the sixth number, $3^2 * 5 = \sqrt[6]{3^2 * 3^3 * 5^3 * 3^2 * 5^2 * 3^5 * N}$

$\sqrt[6]{3^{12} * 5^5 * N}$ Since $\sqrt[6]{3^{12} * 5^6} = 45$, N must be 5.

3.)_ You have a cart containing eight identical items that cost $N each and two identical items that cost $(3N+2) each. If you remove one of each type of item from the cart, the arithmetic mean of the remaining eight items is \$1.50 less than the arithmetic mean of the original ten items. Find the cost of the highest priced item.

$$\frac{8N+2(3N+2)}{10}=1.5+\frac{7N+(3N+2)}{8}$$

$$\frac{14N+4}{10}=1.5+\frac{10N+2}{8}$$

Solve:

$$40\left(\frac{14N+4}{10}=1.5+\frac{10N+2}{8}\right)$$

$$56N+16=60+50N+10$$

$$6N+16=70$$

$$6N=54, N=9, 3N+2=3*9+2=\$29$$

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Match 4 Round 2
Algebra 1:
Quadratic
Equations

1.) $\dfrac{3\pm\sqrt{14}}{2}$

2.) $\pm10, \pm17$

3.) $2, \dfrac{-1}{2}$

1.) Solve for x: 4x(x-3)=5.

$$4x(x-3)=5$$

$$4x^2-12x-5=0$$

$$x=\frac{12\pm\sqrt{144-4*4*(-5)}}{2*4}=$$

$$\frac{12\pm\sqrt{144-(-80)}}{8}$$

$$\frac{12\pm\sqrt{224}}{8}=\frac{12\pm\sqrt{4}\sqrt{4}\sqrt{14}}{8}=$$

$$\frac{12\pm4\sqrt{14}}{8}=\frac{3\pm\sqrt{14}}{2}$$

2.)_ For which integer values of k does the equation $4x^2+kx+4=0$ have two distinct rational solutions?

Need $k^2 - 64$ to be a perfect square greater than 0, so you essentially need Pythagorean triples involving 8. The two such Pythagorean triangles are 6-8-10 and 8-15-17, so $k = \pm 10, \pm 17$

3. Find all values of k such that the equation $1.5 - kx(6 + (k+2)x) = 3k$ has exactly one real solution for x.

Need the discriminant equal to zero, so

$$36k^2 - 4k(3k^2 + 4.5k - 3) = 0$$

$$36k^2 - 12k^3 - 18k^2 + 12k = 0$$

$$k(-12k^2 + 18k + 12) = 0$$

$$-6k(2k^2 - 3k - 2) = 0$$

$$-6k(k-2)(2k+1) = 0$$

$$k = 0,\_clearly\_extraneous$$

$$k = 2, k = \frac{-1}{2}$$

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Match 4 Round 3 Geometry: Similarity

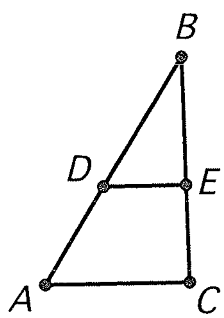
1.) ________ $\frac{20}{3}$ ____________

2.) ______ $\frac{200\sqrt{11}}{11}$ _____cm_

Note: Diagrams are not
Necessarily drawn to scale

3.) __________ $\sqrt{5}$ _________cm

1. ΔABC is a right triangle with the right angle at C. D and E are on $\overline{AB}$ and $\overline{BC}$ respectively. $\overline{DE}$ is parallel to $\overline{AC}$. AC=6, BC=8, and DE= 4. Find BD.



AB = 10 since ΔABC is a 6-8-10 right triangle.

$\frac{DE}{AC} = \frac{BD}{AB}, \frac{4}{6} = \frac{BD}{10},$

6*BD = 40, BD= $\frac{20}{3}$

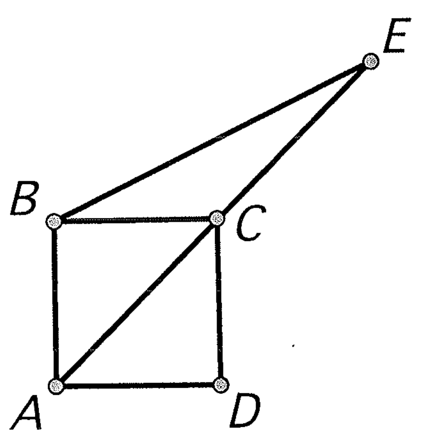
2. The ratio of the areas of two regular decagons is 16:11. One side of the smaller decagon measures 5 cm. Find the perimeter of the larger decagon.

The ratio of the perimeters is the square root of the ratio of their sides, so the ratio

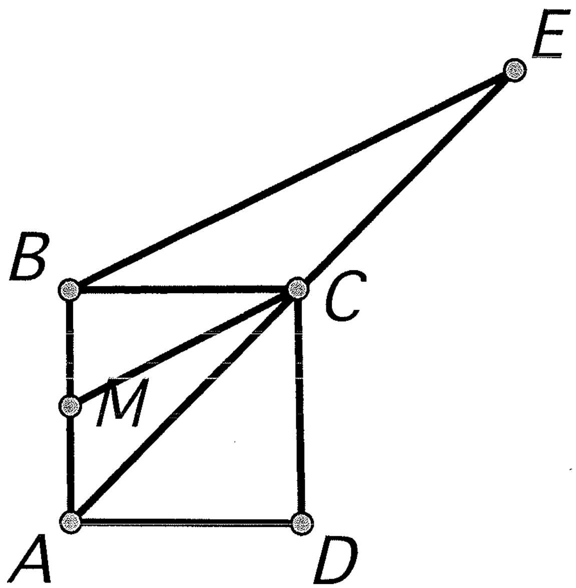
of the perimeters is $\frac{4}{\sqrt{11}}$. The perimeter of the smaller

decagon is 50 cm, so $\frac{4}{\sqrt{11}}=\frac{P}{50}, P=\frac{200}{\sqrt{11}}=\frac{200\sqrt{11}}{11}$,

3. The area of square ABCD is 1 cm^2. Diagonal $\overline{AC}$ is extended to E such that AC = CE. B and E are connected by a line segment. Find BE.



Consider M, the midpoint of $\overline{AB}$.



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In ΔAMC and ΔABE, we have the AE = 2*AC and AB=2*AM, and angle A is the same in both triangles so the triangles are similar by the side-angle-side theorem for similarity (two sides proportional and the included angle congruent). Then

$\frac{BE}{MC} = \frac{AE}{AC} = 2$. MC = $\sqrt{1^2 + (0.5)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$, so BE= 2*MC = $2 * \frac{\sqrt{5}}{2} = \sqrt{5}$ cm

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Match 4 Round 4
Algebra 2:
Variation

1.) $\frac{-2}{5}$

2.) $\pm\frac{3\sqrt{35}}{7}$ ($\frac{3\sqrt{35}}{7}$ was accepted)

3.) 900,000

1.)_ (z+4) varies inversely with the cube root of w. If z=2 when w=27, what is the value of z when w=125?

$(z+4)\sqrt[3]{w}$ is constant, so

$(2+4)\sqrt[3]{27}=(z+4)\sqrt[3]{125}$

$6*3=5(z+4)$

$18=5z+20$

$z=\frac{-2}{5}$

2.)_ m varies directly with n^2, and n varies inversely with p. If p=3 when m=5, what is p when m=7?

$$m = k_1 n^2 \_ n = \pm \frac{k_2}{p},$$

$$so \_ p = \pm \frac{k_2}{n} = \pm \frac{k_2}{\sqrt{\frac{m}{k_1}}} = \pm \frac{K}{\sqrt{m}}$$

$$When \_ m = 5, 3 = \pm \frac{K}{\sqrt{5}}, K = \pm 3\sqrt{5}.$$

$$When \_ m = 7, p = \pm \frac{3\sqrt{5}}{\sqrt{7}} = \pm \frac{3\sqrt{35}}{7}$$

3.) The average number of telephone calls per day between two cities varies jointly with the populations of the two cities and inversely with the square of the distance between them. New York has population 8.5 million and Philadelphia has population 1.6 million. They are 100 miles away from each other. The average number of calls per day between New York and Philadelphia is 27.2 million. The distance between New York and Chicago is 700 miles. The population of Chicago is 2.7 million. What is the average number of phone calls per day between New York and Chicago rounded to the nearest hundred thousand?

Let N = average number of calls between New York and Chicago per day.

$$27.2*10^6 = k\frac{(8.5*10^6)(1.6*10^6)}{100^2}$$

$$k = \frac{(27.2*10^6)100^2}{(8.5*10^6)(1.6*10^6)} = \frac{N*700^2}{(8.5*10^6)(2.7*10^6)}$$

$$N = \frac{(2.7*10^6)(27.2*10^6)(8.5*10^6)*100^2}{700^2(8.5*10^6)(1.6*10^6)} =$$

$$\frac{(2.7*10^6)(27.2*10^6)*100^2}{700^2(1.6*10^6)}$$

Approximate

$$2.7*27.2 \approx 73$$

$$\frac{100^2}{700^2} \approx 0.02$$

$$N = \frac{0.02*73*10^{12}}{1.6*10^6}$$

$$N = \frac{1.46*10^{12}}{1.6*10^6}$$

Approximate $\frac{1.46}{1.6} \approx 0.9$

$$0.9*10^6 = 900,000$$

FAIRFIELD COUNTY MATH LEAGUE 2018-2019

Match 4 Round 5
Trig Expressions and DeMoivre's Theorem

1.) $\frac{3\sqrt{3}-3}{4}$

2.) $(2\sqrt{2})cis(\frac{5\pi}{12})$

3.) 128

1.) Evaluate $\dfrac{\sin(\frac{\pi}{2})+\cos(\frac{\pi}{3})}{\tan(\frac{\pi}{4})+\cot(\frac{\pi}{6})}$

$$\frac{\sin(\frac{\pi}{2})+\cos(\frac{\pi}{3})}{\tan(\frac{\pi}{4})+\cot(\frac{\pi}{6})}$$

$$\frac{\sin(\frac{\pi}{2})+\cos(\frac{\pi}{3})}{\tan(\frac{\pi}{4})+\cot(\frac{\pi}{6})}=\frac{1+\frac{1}{2}}{1+\sqrt{3}}=\frac{\frac{3}{2}}{1+\sqrt{3}}=\frac{3}{2(1+\sqrt{3})}=$$

$$\frac{3(1-\sqrt{3})}{2(1+\sqrt{3})(1-\sqrt{3})}=\frac{(3-3\sqrt{3})}{2(-2)}=\frac{3(1-\sqrt{3})}{-4}=\frac{3\sqrt{3}-3}{4}$$

2.) Find the cube root of -16-16i that is in the first quadrant. Express your answer as r cis(ø) where r is in simplest radical form and ø is in radians.

We have
$$-16-16i=16(-1-i)=16\sqrt{2}(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i)$$
$$=16\sqrt{2}(cis\frac{5\pi}{4})$$

The cube root that is in the first quadrant has $(16\sqrt{2})^{\frac{1}{3}}(cis\frac{5\pi}{12})$

$(16\sqrt{2})^{\frac{1}{3}}=(2^{\frac{9}{2}})^{\frac{1}{3}}=2^{\frac{3}{2}}=2\sqrt{2}$, so the root is $2\sqrt{2}cis(\frac{5\pi}{12})$

$$z^3=-8, so$$
$$z^3=2cis\pi, 2cis(3\pi), 2cis(5\pi)$$
$$z=2cis(\frac{\pi}{3}), 2cis\pi, 2cis(\frac{5\pi}{3})$$
$$w^4=-16, so$$
$$w^4=2cis\pi, 2cis3\pi, 2cis5\pi, 2cis7\pi$$
$$w=2cis(\frac{\pi}{4}), 2cis(\frac{3\pi}{4}), 2cis(\frac{5\pi}{4}), 2cis(\frac{7\pi}{4})$$

If you multiply
$$2cis(\frac{\pi}{3})*2cis(\frac{7\pi}{4})$$
$$4cis(\frac{\pi}{3}+\frac{7\pi}{4})=4cis(\frac{13\pi}{12})$$

which is equivalent to $4cis(\frac{\pi}{12})$

and this is the smallest positive value of ø.

3.) If $(\cos^4(2\o))(\sin(2\o))^2$ is expressed using $\cos(\o)$ as the only trig function, what is the coefficient of $\cos^6(\o)$?

$\cos^4(2\o)(\sin(2\o))^2$

$\cos^4(2\o) = (2\cos^2\o - 1)^4 =$

$(2\cos^2\o - 1)^2 =$

$(4\cos^4\o - 4\cos^2\o + 1)^2 =$

$(4\cos^4\o - 4\cos^2\o + 1)(4\cos^4\o - 4\cos^2\o + 1) =$

$16\cos^8\o - 32\cos^6\o + 24\cos^4\o - 8\cos^2\o + 1$

Then

$(\sin(2\o))^2 = (2\cos\o\sin\o)^2$

$= 4\cos^2(\o) * \sin^2(\o)$

$= (4\cos^2\o)(1 - \cos^2\o)$

$= 4\cos^2\o - 4\cos^4\o$

$\Pr oduct\_is$

$(16\cos^8\o - 32\cos^6\o + 24\cos^4\o - 8\cos^2\o + 1)(4\cos^2\o - 4\cos^4\o)$

$Terms\_with\_\cos^6\o\_are$

$(4\cos^2\o)(24\cos^4\o) + (-8\cos^2\o)(-4\cos^4\o)$

$= 128\cos^6\o$

so_128

FAIRFIELD COUNTY MATH LEAGUE 2018-2019

Match 4 Round 6 Conics

1.) Center: $(-\frac{1}{3},1)$ Radius: $\frac{\sqrt{110}}{3}$

2.) ~~$(\frac{-3}{2},4)$~~ → $(-\frac{15}{8}, 4)$

3.) $-\frac{1}{4}$

1.) Find the center and radius of the circle

$$9x^2+6x+9y^2-18y=100$$

$$9x^2+6x+1+9y^2-18y+9=110$$

$$9(x+\frac{1}{3})^2+9(y-1)^2=110$$

$$Center(-\frac{1}{3},1)$$

$$Radius\_\frac{\sqrt{110}}{3}$$

2.) A parabola of the form $x=ay^2+by+c$ has vertex at (-2,4) and passes through (6,6). Find the focus of the parabola.

Since we know the vertex, the parabola written in vertex form is

$(x-(-2)) = a(y-4)^2$

$x+2 = a(y-4)^2$

Since it passes through (6,6), $(6+2) = a(6-4)^2, 8 = 4a, a = 2$

a=4p, so 2=4p, $p = \frac{1}{2}$, so focus is at $(-2+\frac{1}{2}, 4) = (\frac{-3}{2}, 4)$

$(\frac{-15}{8}, 4)$

3.) A hyperbola has foci at $(2\sqrt{5},0)$ and $(-2\sqrt{5},0)$ and passes through the point $(4\sqrt{2},2)$. Find the product of the slopes of the asymptotes of the hyperbola.

If the x-intercepts are (a,0) and (-a,0) and the y-intercepts are (0,b) and (0,-b), we know a>b since the foci are on the x-axis. Since $(2\sqrt{5})^2 = 20, a^2 + b^2 = 20$. We also have

$$\frac{(4\sqrt{2})^2}{a^2} - \frac{2^2}{b^2} = 1, \frac{32}{a^2} - \frac{4}{b^2} = 1$$

Since $a^2 = 20 - b^2$,

$$\frac{32}{20-b^2} - \frac{4}{b^2} = 1,$$

$$32b^2 + 4b^2 - 80 = -b^4 + 20b^2$$

$$b^4 + 16b^2 - 80 = 0$$

$$(b^2-4)(b^2+20) = 0$$

$$b^2 = 4\_or\_b^2 = -20, not\_possible$$

$$a^2 = 16$$

Asymptotes have slopes $\frac{b}{a}, -\frac{b}{a}$.

$$\frac{b}{a} = \frac{\sqrt{4}}{\sqrt{16}} = \frac{1}{2}$$

$$(\frac{b}{a})(-\frac{b}{a}) = (\frac{1}{2})(\frac{-1}{2}) = \frac{-1}{4}$$

FAIRFIELD COUNTY MATH LEAGUE 2018-2019 Match 4 Tm Rd

1.) _____52.6_______________ 4.) _____ $k = -\frac{1}{4}, n = \frac{4}{3}$ _____

2.) _____ $4x^2 + 2x - 1 = 0$ _____ 5.) _______ $-\frac{28}{85}$ _______

3.) ________2_______________ 6.) __ $\left(\frac{1+\sqrt{7}}{4}, \frac{-1+\sqrt{7}}{2}\right)$ __

1.)_ The inter-quartile range of a set of 5 distinct numbers is the second highest number minus the second lowest number. Give the mean of the set of five consecutive prime numbers that has the largest inter-quartile range if all the prime numbers must be less than 100.

Checking on spreadsheet, given 3 prime numbers less than 100, the greatest possible difference between the first and last numbers is 12. This is when the 3 numbers are 47, 53, and 59. The five numbers are then 43, 47, 53, 59, 61. Their mean is $\frac{43+47+53+59+61}{5} = 52.6$

2) One quadratic equation has solutions $1 \pm \sqrt{5}$. A second quadratic equation has solutions that are the reciprocals of the solutions of the first quadratic equation. Give the second equation in the form y=ax^2 + bx + c = 0 where a,b,c are relatively prime integers and a>0.

The sum of the solutions of the first equation is 2 and their product is $(1+\sqrt{5})(1-\sqrt{5})=-4$, so it has equation $x^2-2x-4=0$. The reciprocals of $1\pm\sqrt{5}$ are

$$\frac{1}{1+\sqrt{5}}\_and\_\frac{1}{1-\sqrt{5}}=$$

$$\frac{1-\sqrt{5}}{(1+\sqrt{5})(1-\sqrt{5})}\_and\_\frac{1+\sqrt{5}}{(1+\sqrt{5})(1-\sqrt{5})}$$

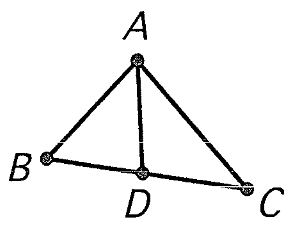
$$=\frac{1-\sqrt{5}}{-4}\_and\_\frac{1+\sqrt{5}}{-4}=\frac{\sqrt{5}-1}{4}\_and\_\frac{-1-\sqrt{5}}{4}$$

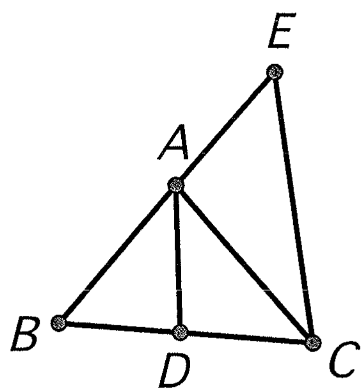
Their sum is $\frac{-1}{2}$ and their product is

$$\frac{(\sqrt{5}-1)(-\sqrt{5}-1)}{4*4}=\frac{-4}{16}=\frac{-1}{4}$$

Let a=4, then $4x^2+2x-1=0$.

3.)_ In the diagram of ΔABC below, not necessarily drawn to scale, $\overline{AD}$ bisects $\angle BAC$. AC=3x+2, DC=x+4, BC=3x+9, AB=7x-2. Give all possible values of x.





Extend $\overline{AB}$ and draw a line from C parallel to $\overline{AD}$. Call E the point where the lines intersect. Since parallel lines are cut by transversal $\overline{AC}$, angles AEC and BAD are congruent by corresponding angles, and angles ACE and DAC are congruent and alternate interior angles. Since $\overline{AD}$ bisects $\angle BAC$, angles BAD and DAC are congruent, so all 4 angles are congruent to each other. This makes ΔAEC isosceles with AC=AE. Since ΔABD is similar to ΔEBC, we have $\frac{AB}{BD} = \frac{BE}{BC}$ and therefore

$\frac{AB}{BD} = \frac{AB + AE}{BD + DC}$, cross multiply to get