# FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016

Match 5 Round 1
Algebra I:
Fractions and
Exponents

1.) ______3.14159______

2.) _______1296_________

3.) _____ $96k^7p$ ________

1) Express the following as a decimal, correct to 5 decimal places:

$$3+\cfrac{1}{7+\cfrac{1}{16}}$$

$3+\cfrac{1}{7+\cfrac{1}{16}}=3+\cfrac{1}{\frac{113}{16}}=3+\frac{16}{113}=\frac{355}{113}$. This is the famous approximation of $\pi$ given determined by both Archimedes and Chinese mathematicians, and it is 3.14159.

2) Express as an integer or a reduced fraction:

$$(36)^5(18)^{-4}(12)^3(6)^{-2}(3)^1(2)^{-6}$$

$$(36)^5(18)^{-4}(12)^3(6)^{-2}(3)^1(2)^{-6} =$$
$$2^5 2^5 3^5 3^5 2^{-4} 3^{-4} 3^{-4} 2^3 2^3 3^3 2^{-2} 3^{-2} 3^1 2^{-6} =$$
$$2^{10-4+6-2-6} 3^{10-8+3-2+1} = 2^4 3^4 = 1296$$

3.) If $k\neq 0$ and $p\neq 0$, simplify the following as much as possible. Do not leave any negative exponents in your answer:

$$\frac{(4k^2)^{2p+1}(3p)^{2-3k}}{(2k)^{4p-3}(27p^3)^{-k+1}}(3kp)^2$$

$$\frac{(4k^2)^{2p+1}(3p)^{2-3k}}{(2k)^{4p-3}(27p^3)^{-k+1}}(3kp)^2$$

$$\frac{4^{2p+1}k^{4p+2}3^{2-3k}p^{2-3k}}{2^{4p-3}k^{4p-3}27^{-k+1}p^{-3k+3}}(9k^2p^2)$$

$$=\frac{(16^p)(4)(k^{4p})(k^2)(9)(27^{-k})p^2p^{-3k}}{16^p(2^{-3})(k^{4p})k^{-3}(27^{-k})(27)(p^{-3k})p^3}(9k^2p^2)$$

$$=\frac{(4k^2)(9)p^2}{(\frac{1}{8}k^{-3})(27)p^3}(9k^2p^2)$$

$$=\frac{32k^5}{3p}(9k^2p^2)=96k^7p$$

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# FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016

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|---|
| Match 5 Round 2<br>Algebra I:<br>Fractional<br>Expressions and<br>Equations |

1.) $\dfrac{2(2x+3)}{3(x+3)}$ or $\dfrac{4x+6}{3x+9}$

2.) ________ $-2, 1$ ________

3.) ___ $\dfrac{y^4+3xy^2+x^2}{y^3+2xy}$ ___

1). Simplify the product as much as possible if no values of x make any denominators equal to zero: $\dfrac{x^2+7x-30}{3x^2-27} * \dfrac{8x^2+24x+18}{2x^2+23x+30}$

$$\frac{x^2+7x-30}{3x^2-27} * \frac{8x^2+24x+18}{2x^2+23x+30} =$$

$$\frac{(x+10)(x-3)}{3(x+3)(x-3)} * \frac{2(2x+3)(2x+3)}{(2x+3)(x+10)} =$$

$$\frac{2(2x+3)}{3(x+3)}$$

2). Solve for all possible values of x:

$$4 - \frac{9}{x+5} = \frac{3x+7}{x+3}$$

$$4-\frac{9}{x+5}=\frac{3x+7}{x+3}$$

$$4(x+5)(x+3)-9(x+3)=(3x+7)(x+5)$$

$$4x^2+32x+60-9x-27=3x^2+22x+35$$

$$x^2+23x+33=3x^2+22x+35$$

$$x^2+x-2=0$$

$$(x+2)(x-1)=0$$

$$x=-2\_or\_x=1$$

3). Simplify as much as possible given that x>0 and y>0. Express your answer as a single fraction.

$$y+\cfrac{x}{y+\cfrac{x}{y+\cfrac{x}{y}}}$$

$$y+\frac{x}{y+\frac{x}{y+\frac{x}{y}}}=$$

$$y+\frac{x}{y+\frac{x}{\frac{y^2+x}{y}}}=$$

$$y+\frac{x}{y+\frac{xy}{y^2+x}}=$$

$$y+\frac{x}{\frac{y^3+2xy}{y^2+x}}=$$

$$y+\frac{xy^2+x^2}{y^3+2xy}=$$

$$\frac{y^4+2xy^2+xy^2+x^2}{y^3+2xy}=$$

$$\frac{y^4+3xy^2+x^2}{y^3+2xy}$$

**FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016**

Match 5 Round 3
Geometry:
Circles

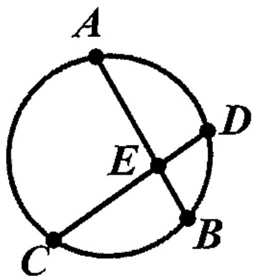
1.) ________11__________

2.) __________$225\pi$___$cm^2$

Note: Diagrams not necessarily to scale.

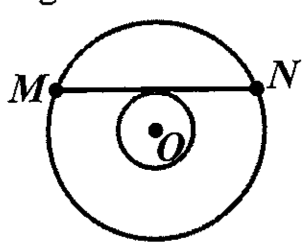
3.) __________ $4\sqrt{6}$ __________cm

1). Points A, B, C, and D lie on a circle. $\overline{AB}$ intersects $\overline{CD}$ at point E. AE=6, EB=4, CE=x, DE=x-5. Find the length of $\overline{CD}$.



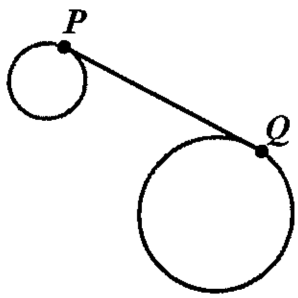

AE*EB = CE*DE. 6*4=x(x-5). $x^2 - 5x - 24=0$. (x-8)(x+3)=0. x=8, so CE=8, DE=3.
CD=11.

2). Two circles are concentric. Points M and N lie on the larger circle and $\overline{MN}$ is tangent to the smaller circle. If MN = 30 cm, find the area between the two circles.

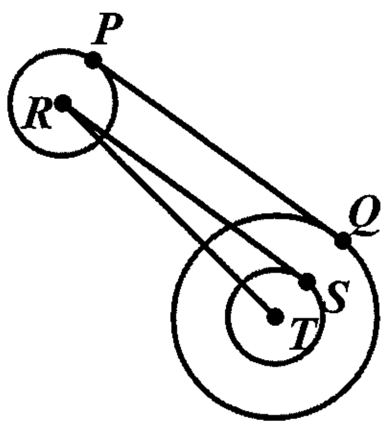



Let r=radius of small circle. R = radius of larger circle. Draw the radius of the smaller circle perpendicular to $\overline{MN}$, say they meet at point K. Then $\Delta$OKN forms a right triangle, with $15^2 = R^2 - r^2$. The desired area is $\pi R^2 - \pi r^2 = \pi (R^2 - r^2)$. Since $R^2 - r^2 = 225$, the desired area is $225\pi$.

3.) A segment is tangent to two circles as shown at points P and Q. The radius of the smaller circle is 2 cm and the radius of the larger circle is 4 cm. The distance between the two centers is 10 cm. Find the length of $\overline{PQ}$




Draw a circle of radius 2 inside the larger circle with the same center as the larger circle.




Then PQRS forms a rectangle, so RS is the same as PQ. ΔRST is a right triangle. Since RT=10 and ST=2, RS= $\sqrt{10^2 - 2^2} = 4\sqrt{6}$. So PQ = $4\sqrt{6}$

## FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016

Match 5 Round 4
Quadratic Equations and Complex Numbers

1.) -64

2.) 169

3.) $\frac{-i}{4}, \frac{1}{3}$

1) Simplify: $\frac{(i-3i^3)^3}{i^{33}}$

$$\frac{(i-3i^3)^3}{i^{33}}$$

$$3i^3 = -3i$$

$$i-(-3i) = 4i$$

$$(4i)^3 = 64i^3 = -64i$$

$$i^{33} = (i^4)^8 i = 1^8 i = i, so$$

$$\frac{-64i}{i} = -64$$

2) $a+bi = \frac{(5+12i)}{(3+2i)}(3-2i)^3$. If a+bi is plotted on the complex plane, how far is it from the origin?

This is easy with DeMoivre's theorem, but since this is an Algebra 2 round, divide $\frac{5+12i}{3+2i} = \frac{(5+12i)(3-2i)}{(3+2i)(3-2i)} = \frac{39+26i}{13} = 3+2i$

$(3+2i)(3-2i)^3 = (3+2i)(3-2i)(3-2i)^2$
$= 13(5-12i)$
$|5-12i| = 13, so\_13*13 = 169$

3) Solve for all complex z: $12iz^2 - (3+4i)z + 1 = 0$

$$z = \frac{(3+4i) \pm \sqrt{(3+4i)^2 - 4*12i*1}}{2*12i}$$

$$= \frac{(3+4i) \pm \sqrt{-7+24i-48i}}{24i}$$

$$= \frac{(3+4i) \pm \sqrt{-7-24i}}{24i}$$

To find $\sqrt{-7-24i}$, set $(a+bi)^2 = -7-24i$
$a^2 - b^2 = -7$
2ab=-24, so a=3, b=-4 or a=-3, b=4. Since we have the plus or minus anyway, that takes care of both possibilities.

$$= \frac{(3+4i) \pm (3-4i)}{24i}\_or\_\frac{(3+4i) \pm (3-4i)}{24i}$$

$$= \frac{6}{24i}\_or\_\frac{8i}{24i}$$

$$= \frac{-i}{4}\_or\_\frac{1}{3}$$

## FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016

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|---|
| Match 5 Round 5<br>Solving Trig<br>Equations |

1.) ______ $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ ______

2.) ______ $-\frac{5}{9}$ ______

3.) ______ $-\frac{\pi}{6}, \frac{\pi}{6}$ ______

1) Solve for all x if $0 \leq x < 2\pi$: $\cos(4x) = -1$

$4x = \pi$ or $3\pi$ or $5\pi$ or $7\pi$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

2) If $\cos(x) + \sin(x) = \frac{2}{3}$, give the value of $\sin(2x)$.

Square both sides.

$$\cos^2 x + 2\sin x\cos x + \sin^2 x = \frac{4}{9}$$

$$1 + 2\sin x\cos x = \frac{4}{9}$$

$$2\sin x\cos x = -\frac{5}{9}$$

$$\sin(2x) = -\frac{5}{9}$$

3) Solve for x if $-\frac{\pi}{2} < x < \frac{\pi}{2}$: $3\sec(x)\tan^2(x) - \sec(x) + \frac{3}{\cos^2(x)} = 4$

$$3\sec(x)\tan^2(x) - \sec(x) + \frac{3}{\cos^2(x)} = 4$$

$$3\sec(x)(\sec^2(x) - 1) - \sec(x) + 3\sec^2(x) - 4 = 0$$

$$3\sec^3(x) - 3\sec(x) - \sec(x) + 3\sec^2(x) - 4 = 0$$

$$3\sec^3(x) - 4\sec(x) + 3\sec^2(x) - 4 = 0$$

$$\sec(x)(3\sec^2(x) - 4) + 1(3\sec^2(x) - 4) = 0$$

$$(\sec(x) + 1)(3\sec^2(x) - 4) = 0$$

$$\sec(x) = -1\_or\_\sec^2(x) = \frac{4}{3}$$

$$\cos(x) = -1\_or\_\cos^2(x) = \frac{3}{4}$$

$$\cos(x) = -1\_or\_\cos(x) = \pm\frac{\sqrt{3}}{2}$$

cos(x)=-1, $\cos(x) = -\frac{\sqrt{3}}{2}$ have no solution in the interval.

$\cos(x) = \frac{\sqrt{3}}{2}$ at $-\frac{\pi}{6}, \frac{\pi}{6}$

# FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016

Match 5 Round 6
Sequences and
Series

1.) ____________204________

2.) ______$\frac{3}{8}$__________

3.) _________3, −59__________

1.) Evaluate $\sum_{n=1}^{8} n^2$.

1+4+9+16+25+36+49+64=204

2.) For a geometric sequence {$a_n$}, $\sum_{n=1}^{\infty} a_n = 4$. If $a_2 = -3$ , what are all possible values for the fifth term of the sequence?

$$4 = \frac{a_1}{1-r}, \_a_2 = a_1 r = -3, \_so\_a_1 = \frac{-3}{r}$$

$$4 = \frac{-3}{r(1-r)}$$

$$4r - 4r^2 = -3$$

$$0 = 4r^2 - 4r - 3$$

$$0 = (2r+1)(2r-3)$$

$$r = \frac{-1}{2}, \frac{3}{2}$$

but if r= $\frac{3}{2}$, the series does not converge, so $a_1 = 6, a_5 = 6(-\frac{1}{2})^4 = \frac{3}{8}$

3.) In an arithmetic sequence, the twelfth term is 30 less than the square of the second term. If the third term is 19, find all possible values for the first term of the sequence.

$$a_1 + 2d = 19,\_so\_a_1 = 19 - 2d$$

$$(a_1 + 11d) = (a_1 + d)^2 - 30$$

$$(19 - 2d + 11d) = (19 - 2d + d)^2 - 30$$

$$(19 + 9d) = (19 - d)^2 - 30$$

$$19 + 9d = 361 - 38d + d^2 - 30$$

$$d^2 - 47d + 312 = 0$$

$$(d - 8)(d - 39) = 0$$

$$d = 8\_or\_d = 39$$

$$a_1 = 19 - 2 * 8\_or\_19 - 2 * 39$$

$$a_1 = 3, -59$$

## FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016

Match 5 Team Round

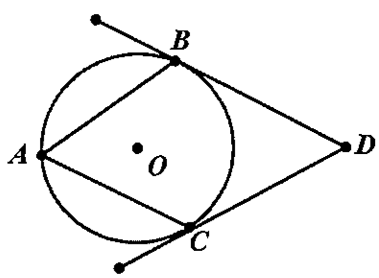
1.) ____$16\sqrt{3}$____ 4.) _$13-9i,-13+9i$___

2.) __$16a^3b-16ab^3$_ 5.)____ $\frac{-4\pm3\sqrt{3}}{10}$ _____

Note: Diagrams not necessarily drawn to scale.

3.) ________$3,-\frac{3}{4}$_____ 6.)________101, -89___________

1) In the figure below, circle O has radius 4 cm. is inscribed in the circle with center O. Arc BAC measures 240 degrees. $\overline{BD}$ and $\overline{CD}$ are tangent to circle O. Find the perimeter of quadrilateral ABDC.




Draw segments $\overline{OA},\overline{OB},\overline{OC},\overline{OD}$. Since arc BAC is 240 degrees, the rest of the circle is 120 degrees, and since $\angle BOA, \angle COA$ is an inscribed angle, it measures 60 degrees. Since AB=AC, OA=OB, and $\angle BAO, \angle OAC$ are each 30 degrees. OA=OB=OC since all are radii, so $\angle BOA, \angle COA$ are each 120 degrees. Consider $\Delta$AOB. By law of cosines, find AB.

$$(AB)^2 = 4^2 + 4^2 - 2*4*4*\cos(120)$$

$$(AB)^2 = 32 - 32(\frac{-1}{2}) = 48$$

$$AB = \sqrt{48} = 4\sqrt{3}$$

DB=DC since both are tangents. $\angle CDB$ = 60 degrees by $\frac{240-120}{2}$. $\Delta$ODB and $\Delta$ODC are both 30-60-90 triangles with short side 4 cm, so the long side is $4\sqrt{3}$ cm, so BD=DC=$4\sqrt{3}$

The total perimeter is $4(4\sqrt{3}) = 16\sqrt{3}$ $4(4\sqrt{3}) = 16\sqrt{3}$

.) Simplify as much as possible: $$\frac{\frac{a^2-b^2}{ab^3}-\frac{b^2-a^2}{a^3b}}{\frac{a^2+b^2}{(2ab)^4}}$$

$$\frac{\frac{a^2-b^2}{ab^3}-\frac{b^2-a^2}{a^3b}}{\frac{a^2+b^2}{(2ab)^4}}=$$

$$\frac{\frac{a^2(a^2-b^2)}{a^3b^3}-\frac{b^2(b^2-a^2)}{a^3b^3}}{\frac{a^2+b^2}{(2ab)^4}}=$$

$$\frac{\frac{a^4-b^4}{a^3b^3}}{\frac{a^2+b^2}{(2ab)^4}}=\frac{\frac{(a^2-b^2)(a^2+b^2)}{a^3b^3}}{\frac{a^2+b^2}{(2ab)^4}}=$$

$$\frac{(a^2-b^2)(16a^4b^4)}{a^3b^3}=16a^3b-16ab^3$$

3.) Solve this equation for all possible values of x: $$2+\frac{3}{2+\frac{3}{x}}=\frac{3}{2-\frac{3}{x}}$$

$$2+\frac{3}{\frac{2x+3}{x}}=\frac{3}{\frac{2x-3}{x}}$$

$$2+\frac{3x}{2x+3}=\frac{3x}{2x-3}$$

$$2(2x+3)(2x-3)+3x(2x-3)=3x(2x+3)$$

$$8x^2-18+6x^2-9x=6x^2+9x$$

$$8x^2-18x-18=0$$

$$2(4x+3)(x-3)=0$$

$$x=\frac{-3}{4},3$$

4.) A geometric sequence $\{a_n\}$ of complex numbers has $a_1=1+i$ and $a_3=7-i$. Find all possible values for $a_4$.

$$r^2=\frac{a_3}{a_1}=\frac{7-i}{1+i}=\frac{(7-i)(1-i)}{(1+i)(1-i)}=\frac{6-8i}{2}=3-4i$$

$$r=\sqrt{3-4i}=a+bi$$

$$3-4i=a^2-b^2+2abi$$

$$a=2\_and\_b=-1\_or\_a=-2\_and\_b=1$$

$$r=a+bi=2-i\_or\_-2+i$$

$$a_4=(7-i)(2-i)=13-9i$$

$$or\_a_4=(7-i)(-2+i)=-13+9i$$

5.) Find all values of sin(A) such that $\cos(A+\frac{\pi}{6})=\frac{4}{5}$

$$\cos(A+\frac{\pi}{6})=\frac{4}{5}$$

$$\cos A\cos\frac{\pi}{6}-\sin A\sin\frac{\pi}{6}=\frac{4}{5}$$

$$\cos A(\frac{\sqrt{3}}{2})-\sin A(\frac{1}{2})=\frac{4}{5}$$

$$5\cos A\sqrt{3}-5\sin A=8$$

$$5\cos A\sqrt{3}=5\sin A+8$$

Square both sides and use $\cos^2 A = 1 - \sin^2 A$

$$75\cos^2 A=25\sin^2 A+80\sin A+64$$

$$75(1-\sin^2 A)=25\sin^2 A+80\sin A+64$$

$$100\sin^2 A+80\sin A-11=0$$

$$\sin A=\frac{-80\pm\sqrt{6400-(-4400)}}{200}=\frac{-80\pm\sqrt{10800}}{200}=\frac{-80\pm10\sqrt{108}}{200}=\frac{-80\pm60\sqrt{3}}{200}=$$

$$\frac{-4\pm3\sqrt{3}}{10}$$

6) An arithmetic sequence of real numbers has as its first 3 terms: a, 2a-1, $a^2 - 90$. Find all possible values of for the $10^{th}$ term of the sequence.

(2a-1)-a=d, so d=a-1. a+2d=$a^2$-90, a+2(a-1)=$a^2$ – 90. $a^2$ -3a – 88=0

(a-11)(a+8)=0. a=11 or a=-8

If a=11, d=10, so 11+9*10=101

If a=-8, d=-9, so -8+9*(-9)=-89