FAIRFIELD COUNTY MATH LEAGUE (FCML)2015-2016

Match 4 Round 1
Arithmetic:
Basic Statistics

1.) _______23.25_________

2.) ________36________

3.) _______18__________

1.)Find the arithmetic mean of the median and the mode of the numbers {13, 23, 24, 24, 15, 18, 24, 22}.

The mode is 24, and the median is 22.5. The mean of these two numbers is (24+22.5)/2 = 23.25

2.) The geometric mean of the numbers $x_1, x_2, x_3, ... x_n$ is defined to be $\sqrt[n]{x_1 x_2 x_3 ... x_n}$. Find the geometric mean of the set of numbers {8, 12, 36, 72, 243}

Putting these into prime factors, we get 2^3, 2^2*3, 2^2*3^2, 2^3*3^2, and 3^5, so they multiply to $2^{10}3^{10} = 6^{10}$, and $\sqrt[5]{6^{10}} = 36$

3.) {2,3,5} is a set of consecutive prime numbers whose median is equal to its range. There is one other set of consecutive prime numbers, all of which are between 5 and 100, such that the range of the numbers is equal to the median of the numbers. Find this number that is equal to the range and the median of this set.

The only set of consecutive primes that meets the criteria is {11,13,17,19,23,29} (verified by spreadsheet). It has range 18 and median 18. Since the average distance between primes increases as the primes increase, if you increase the range, the median will be smaller than the range.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016

Match 4 Round 2 Algebra 1: Quadratic Equations

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1.) ________0, 6___________

2.) _____________9, -16_________

3.) ________ $5, \dfrac{5}{m-1}$ ___________

1.)Find all solutions to the equation $(x+3)^2 = (2x-3)^2$

$x^2 + 6x + 9 = 4x^2 - 12x + 9$

$3x^2 - 18x=0$

$3x(x-6)=0$

$x=0$ or $x=6$.

2.) Find all values of k such that $(k+7)x^2 - 24x + k=0$ has exactly one real solution.

The discriminant must be zero, so $24^2 - 4k(k+7)=0$

$576=4k(k+7)$

$144=k(k+7)$

$k^2 + 7k - 144=0$

$(k+16)(k-9)=0$

$k=-16$ or $k=9$

3.)_If $m \neq 1$, give all solutions to the equation

$(m-1)x^2 + 25= 5mx$. Express your answers as single fractions involving m.

$(m-1)x^2 - 5mx + 25 = 0$.

By quadratic formula,

$$x = \frac{5m \pm \sqrt{(5m)^2 - 4(25(m-1))}}{2(m-1)}$$

$$= \frac{5m \pm \sqrt{(25m^2 - 100m + 100)}}{2(m-1)}$$

$$= \frac{5m \pm 5(m-2)}{2(m-1)} = \frac{10m - 10}{2(m-1)} = 5\_ or \frac{10}{2(m-1)} = \frac{5}{m-1}$$

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FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016

Match 4 Round 3
Geometry:
Similarity

Note: Diagrams are not
Necessarily drawn to scale

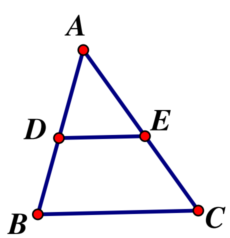
1.) ________4.8___________

2.) ________ $\frac{96\sqrt{5}}{5}$ ___________cm__

3.) __________135_____________

1. In the diagram below, $\overline{BC}$ is parallel to $\overline{DE}$. AD=6, DB=4,

BC=8. Find DE.



ΔADE is similar to ΔABC, so

$$\frac{AD}{AB} = \frac{DE}{BC}$$
$$\frac{6}{10} = \frac{DE}{8}$$
$$10 * DE = 48$$
$$DE = 4.8$$

2. The ratio of the areas of two regular octagons is 9:5. One side of the smaller octagon measures 4 cm. Find the perimeter of the larger octagon.

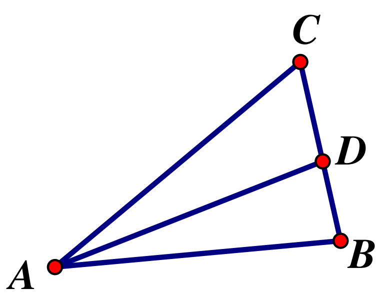
The ratio of the perimeters is the square root of the ratio of their sides, so the ratio of the perimeters is $\frac{3}{\sqrt{5}}$. The perimeter of the smaller octagon is 32 cm, so

$$\frac{3}{\sqrt{5}} = \frac{p}{32}$$

$$p\sqrt{5} = 96$$

$$p = \frac{96}{\sqrt{5}} = \frac{96\sqrt{5}}{5}$$

3. In ΔABC below, D lies on $\overline{BC}$, $\overrightarrow{AD}$ bisects $\angle BAC$, AC=40 , AB=24 , and AB=BC. Find the product of DB and DC.



By the angle bisector theorem, $\frac{AB}{BD} = \frac{AC}{DC}$. Let DB=x. Since AB=BC, BC=24. Then DC=24-x. So

$$\frac{40}{24-x} = \frac{24}{x}$$

$$40x = 576 - 24x$$

$$64x = 576$$

$$x = 9$$

$$24 - x = 15$$

The product of the two segments is 135.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016

Match 4 Round 4
Algebra 2:
Variation

1.) ______9_____________

2.) _________500_____pounds

3.)__k=__1.5______n=__3_______

1.) z varies inversely with the cube root of (y+3). If z=6 when y=24, what is the value of z when y=5?

$z\sqrt[3]{y+3}$ is constant.

$6\sqrt[3]{24+3} = z\sqrt[3]{5+3}$

$6*3 = z*2$

$z = 9$

2._ The safe load a beam can support varies jointly with the width and the square of the depth and inversely as the length. If a beam measuring 16 feet by 2 inches by 8 inches is positioned so that the width is 2 inches and the depth is 8 inches, it can support 2,000 pounds. How much weight can the same beam support if it is turned so that the width is 8 inches and the depth is 2 inches?

Let S=safe load in pounds, L = length in feet, W=width in inches, and D=depth in inches

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$$S = k\frac{WD^2}{L}$$

$$2000 = k\frac{2*8^2}{16}, k = 250$$

$$S = 250\frac{8*2^2}{16} = 500$$

3. A direct power variation function has form $y=kx^n$. Both (4, 96) and (9, 324) belong to the function. Find the values of k and n.

$$324 = k * 6^n$$

$$96 = k * 4^n$$

Divide these to get $\frac{324}{96} = (\frac{6}{4})^n$, $(\frac{3}{2})^n = \frac{27}{8}$

so n=3.

$96=k*4^3$, so 64k=96, k=1.5

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016

Match 4 Round 5
Trig Expressions

1.) _________ -1___________

2.) ______ $-128+(128\sqrt{3})i$_______

3.) ____________-4 tan x_________

1.) Simplify as much as possible given (cos x)≠0 and (sin x)≠0 and sin(x)≠1 or sin(x)≠-1:

$$\frac{1-\cos^2 x}{\sin^2 x-1}*\frac{\cot^2 x+1}{\tan^2 x+1}$$

$$\frac{1-\cos^2 x}{\sin^2 x-1}*\frac{\cot^2 x+1}{\tan^2 x+1}=$$

$$\frac{1-\cos^2 x}{\sin^2 x-1}*\frac{\csc^2 x}{\sec^2 x}$$

$$=\frac{\sin^2 x}{-\cos^2 x}*\frac{\frac{1}{\sin^2 x}}{\frac{1}{\cos^2 x}}=-1$$

2.) Express $(1+i\sqrt{3})^8$ as a complex number a+bi . (Suggestion: Use DeMoivre's theorem)

$1+i\sqrt{3}$ expressed in r cis ø form is 2 cis $\frac{p}{3}$. $(1+i\sqrt{3})^8$ is 2^8 cis $\frac{8p}{3}$ =

256 cis $\frac{2p}{3}$ = $256(-\frac{1}{2}+i\frac{\sqrt{3}}{2})$ = $-128+(128\sqrt{3})i$ (Acceptable to write $-128+128i\sqrt{3}$)

3.) Express as simply as possible in terms of tan x, given (cos x)≠0 and (sin x)≠0 and cos(2x)≠0:

$$(\tan x+\cot x)(\sin(2x))(\tan(2x)(\tan x+1)(\tan x-1)$$

$$(\tan x+\cot x)(\sin(2x))(\tan(2x)(\tan x+1)(\tan x-1)$$

$$=(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x})(2\sin x\cos x)\frac{2\tan x}{1-\tan^2 x}(\tan x+1)(\tan x-1)$$

$$=(\frac{\sin^2 x+\cos^2 x}{\cos x\sin x})(2\sin x\cos x)\frac{2\tan x}{(1+\tan x)(1-\tan x)}(\tan x+1)(\tan x-1)$$

$$=(\frac{1}{\cos x\sin x})(2\sin x\cos x)=2*(-2\tan x)=-4\tan x$$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016

Match 4 Round 6
Conics

1.) Center: (__-1_,_2__) Radius: ___$\sqrt{3}$_____

2.__(____3____, __$2+\sqrt{7}$_)___

3.)___(4,6), (4,-6), $(\frac{-34}{9},\frac{17}{3})$, $(\frac{-34}{9},\frac{-17}{3})$_________

1.) Find the center and radius of the circle $3x^2 + 6x + 3y^2 – 12y + 6 = 0$
$3(x^2 +2x) + 3(y^2 -4y) + 6=0$
$3(x^2 + 2x+1) + 3(y^2-4y+4)+6=3+12$
$3(x+1)^2 + 3(y-2)^2 = 9$
$(x+1)^2 + (y-2)^2 = 3$
Center (-1, 2) Radius $\sqrt{3}$

2.)An ellipse with major and minor axes parallel to the lines x=0 and y=0 centered at (3,2) has passes through (3,6) and (6,2). Find the coordinates of the focus that is farthest away from the origin.

The semi-major axis must be along the y-axis and has length 4, which is 6-2. The semi-minor axis must be along the x-axis and has length 3, which is 6-3. The equation of the ellipse is $\frac{(x-3)^2}{9}+\frac{(y-2)^2}{16}=1$. The foci are along the line x=3. The distance from the center of the ellipse is to each focus is $\sqrt{16-9}=\sqrt{7}$, so the focus farthest from the axis has coordinates $(3,2+\sqrt{7})$

3.) Give the coordinates of all four points where the parabola with vertex (-68,0) that passes through the points (-50,3) and (-18,-5) intersects the two asymptotes of the hyperbola $9x^2-4y^2=36$

The parabola must open to the right, since one point is above the y-axis and one point is below. Substituting either point into $x=ay^2 – 68$, say (-50,3), gives $-50=a*3^2 – 68$, so 9a=18, and a=2, so the equation is $x=2y^2 –68$. The hyperbola has equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$, so the asymptotes have equations $y = \frac{3}{2}x$ and $y = \frac{-3}{2}x$. Solving x=2*($\frac{3}{2}x$)^2 -68 gives $x=4.5x^2 - 68$, so $9x^2 - 2x - 136=0$, so (x-4)(9x+34)=0.

x=4 or x=$\frac{-34}{9}$.

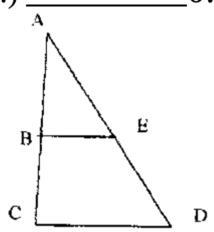
If x=4, y=6 or -6. If x=$\frac{-34}{9}$, y = $\frac{17}{3}$ or $\frac{-17}{3}$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2015-2016 Match 4 Tm Round
Diagrams not necessarily drawn to scale.

1.) ______ $\frac{85}{3}$__________ 4.) ________25 _________

2.) _________1.5____________ 5.) ________16__________

3.) __________0.6______ 6.) ___$12x^2 - 13x + 3 = 0$_____



1.)_ In the figure above $\overline{BE}$ is parallel to $\overline{CD}$. If BE=8, DE=x-6, AD=2x-10, AB=x-7, and BC=0.2x. What are the possible values for the perimeter of ΔACD?

$$\frac{AB}{AC} = \frac{AE}{AD}$$

$$\frac{x-7}{x-7+0.2x} = \frac{(2x-10)-(x-6)}{2x-10}$$

$$\frac{x-7}{1.2x-7} = \frac{x-4}{2x-10}$$

$$(1.2x-7)(x-4) = (x-7)(2x-10)$$

$$1.2x^2 - 11.8x + 28 = 2x^2 - 24x + 70$$

$$0.8x^2 - 12.2x + 42 = 0$$

$$4x^2 - 61x + 210 = 0$$

$$(x-10)(4x-21) = 0$$

$$x = 10\_or\_x = \frac{21}{4}$$

Reject x= $\frac{21}{4}$ because it would lead to negative values for several sides.

If x=10, DE=4, AD=10, AB=3, BC=2. Then $\frac{AE}{BE} = \frac{AD}{CD}, \frac{6}{8} = \frac{10}{CD}, CD = \frac{40}{3}$,

perimeter is 10+5+$\frac{40}{3}$= $\frac{85}{3}$

2.) The number of eggs laid is directly proportional to the number of chickens and inversely proportional to the time. If a chicken-and-half lays an egg-and-a-half in a day-and-a-half, how many eggs do 2 chickens lay in 2 days?

Let E=eggs laid, N=number of chickens, and T= time. The variation is $E = k\frac{N}{T}$

$$1.5 = k\frac{1.5}{1.5}, k = 1.5$$

$$E = 1.5 * \frac{2}{2}, E = 1.5$$

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3.) The inter-quartile range of a set of numbers is the difference between the upper quartile (the median of the upper half of the data) and the lower quartile (the median of the lower half of the data). If there are two data points at the median value, one of them may be considered in either the lower half or the upper half of the data. A direct second power variation function has equation $f(x)=kx^2$ for some value of k and passes through the point (10,4). Find the inter-quartile range of the numbers {f(-4), f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), f(4), f(5)}

$f(x)=kx^2$. $4=k(10^2)$, so k=0.04. $f(x) = 0.04x^2$
The smallest is f(0) = 0. f(1) = f(-1)=0.04, f(2)=f(-2) = 0.16. f(3)=f(-3) = 0.36, f(4)=f(-4)=0.64, and f(5)=1. In order, the numbers are 0, 0.04, 0.04, 0.16, 0.16, 0.36, 0.36, 0.64, 0.64, 1. The lower half of the data is 0, 0.04, 0.04, 0.16, and 0.16. The upper half of the data is 0.36, 0.36, 0.64, 0.64, 1. The median of the lower half is 0.04. The median of the upper half is 0.64. The interquartile range is 0.64 – 0.04 = 0.6.

4. Find the area of the triangle created by the two foci of the hyperbola $\frac{(x-1)^2}{16} - \frac{y^2}{9} = 1$ and the vertex of the parabola $y = 2x^2 - 4x + 7$.

The hyperbola is centered at (1,0). The foci will be on the x-axis. The distance from the center to each focus is $\sqrt{16+9} = 5$, so the foci are at (6,0) and (-4,0). Find the vertex of $y = 2x^2 - 4x + 7$ by $y = 2(x^2 - 2x + 1) + 7 - 2$, so the vertex is at (1,5). The line containing (1,5) and (6,0) is y=-x+6, and the line containing (1,5) and (-4,0) is y=x+4, so the lines have slopes that are opposite reciprocals and are perpendicular. The length of each of the two equal legs of the right triangle is $\sqrt{(6-1)^2 + (0-5)^2} = 5\sqrt{2}$. The area of the triangle is $\frac{1}{2}(5\sqrt{2})(5\sqrt{2}) = 25$

5. If sin(4x)≠0, what is the coefficient of $\cos^4 x$ when the expression $\frac{\sin(8x)}{\sin(4x)}$ is expressed in terms of cos(x)?

$$\frac{\sin(8x)}{\sin(4x)} = \frac{2\cos(4x)\sin(4x)}{\sin(4x)} = 2\cos(4x)$$

$$2\cos(4x) = 2(2\cos^2(2x) - 1) = 4\cos^2(2x) - 2$$

$$4\cos^2(2x) - 2 = 4(2\cos^2(x) - 1)^2 - 2$$

$$= 16\cos^4(x) - 16\cos^2(x) + 2$$

The coefficient is 16.

6. Find a quadratic equation whose solutions are the positive square roots of the solutions of $144x^2 - 97x + 9 = 0$. Express your answer as $ax^2 + bx + c=0$, where a,b, and c are relatively prime integers and a>0.

The equation factors to (9x-1)(16x-9), so $x = \frac{1}{9} or\_x = \frac{9}{16}$. The positive square roots are $\frac{1}{3}\_and\_\frac{3}{4}$.

$$(x - \frac{1}{3})(x - \frac{3}{4}) = 0$$

$$x^2 - \frac{13}{12}x + \frac{3}{12} = 0$$

$$12x^2 - 13x + 3 = 0$$