FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 5 Round 1
Algebra I:
Fractions and
Exponents

1.) $\frac{5}{2}$

2.) $\frac{125}{a^9b^{48}}$

3.) $\frac{m^2-k^2}{k^5m^5}$

1) Express as an integer or a reduced fraction:

$$(100)^2(50)^{-3}(25)^{-4}(10)^5(5)^6(2)^{-7}$$
$$=\frac{10^9*5^6}{50^3 25^4 2^7}=\frac{10^9*5^6}{(5*10)^3(5^2)^4 2^7}=\frac{10^6*5^6}{5^{11}*2^7}=\frac{2^6*5^6}{5^5*2^7}=\frac{5}{2}$$

2) Simplify as much as much as possible. Write your coefficient as a decimal number multiplied or divided by powers of variables using no negative exponents:

$$\left(10a^3b^4\right)^{-6}\left[(5a)^3(2b^{-4})^2\right]^3$$
$$=10^{-6}a^{-18}b^{-24}[(125a^3)(4b^{-8})]^3$$
$$=10^{-6}a^{-18}b^{-24}(500^3a^9b^{-24})$$
$$=10^{-6}a^{-9}b^{-48}5^3 10^6$$
$$=\frac{125}{a^9b^{48}}$$

3) Perform the operations and simplify as much as possible, given that k≠0, m≠0, and m≠k or m≠-k. Express as a single fraction with no negative exponents.

$$\frac{\frac{m^2-k^2}{k^5m^3}-\frac{k^2-m^2}{k^3m^5}}{m^2+k^2}=\frac{m^2(m^2-k^2)-k^2(k^2-m^2)}{k^5m^5(m^2+k^2)}=$$

$$=\frac{m^4-k^4}{k^5m^5(m^2+k^2)}=\frac{(m^2+k^2)(m^2-k^2)}{k^5m^5(m^2+k^2)}=\frac{m^2-k^2}{k^5m^5}$$

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Match 5 Round 2
Algebra I:
Fractions and
Exponents

1.) $\frac{-2x+8}{-x+7}$ alternatively $\frac{2x-8}{x-7}$

2.) ___________-1_________

3.) ___ $\frac{7r^2-20r+6}{r^2-10r+25}$ ______

1). Express as the quotient of two simplified polynomials. Express the terms of your polynomials in order of decreasing degree.

$$1-\cfrac{1}{1-\cfrac{2}{1+\cfrac{3}{x-4}}}=$$

$$=1-\cfrac{1}{1-\cfrac{2}{\cfrac{x-1}{x-4}}}=1-\cfrac{1}{1-\cfrac{2(x-4)}{x-1}}=1-\cfrac{1}{\cfrac{x-1-2(x-4)}{x-1}}=1-\cfrac{1}{\cfrac{-x+7}{x-1}}=$$

$$1-\frac{x-1}{-x+7}=\frac{-x+7-(x-1)}{-x+7}=\frac{-2x+8}{-x+7} \text{ or } \frac{2x-8}{x-7}$$

2) Solve for all possible values of x:

$\frac{5x}{x-5}+\frac{4}{x+6}=\frac{54x+5}{x^2+x-30}$ multiply by LCD $(x+6)(x-5)\left[\frac{5x}{x-5}+\frac{4}{x+6}=\frac{54x+5}{x^2+x-30}\right]$

then $5x(x+6)+4(x-5)=54x+5$, then $5x^2+30x+4x-20=54x+5$ then

$5x^2-20x-25=0$, then 5(x-5)(x+1)=0, so x=-1 because x=5 is extraneous

3). Add and subtract these fractions and simplify your answer as much as possible. Assume that no values of r will make any expressions in the problem equal to zero. (Note: the first sign between fractions is division and the second sign is addition).

$$\frac{4r^2-1}{2r^2-9r-5}\div\frac{2r^2-21r+10}{3r^2-31r+10}+\frac{4r^2-4r+1}{r^2-10r+25}=$$

$$\frac{(2r+1)(2r-1)}{(2r+1)(r-5)}*\frac{(3r-1)(r-10)}{(2r-1)(r-10)}+\frac{4r^2-4r+1}{(r-5)(r-5)}=\frac{3r-1}{r-5}+\frac{4r^2-4r+1}{(r-5)(r-5)}=$$

$$\frac{(3r-1)(r-5)}{(r-5)^2}+\frac{4r^2-4r+1}{(r-5)(r-5)}=\frac{3r^2-16r+5+4r^2-4r+1}{(r-5)(r-5)}=\frac{7r^2-20r+6}{r^2-10r+25}$$

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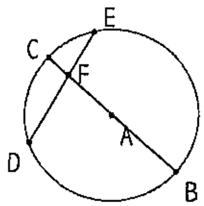
Match 5 Round 3
Geometry:
Circles

1.) _______14 and 4______

2.) ____240 (degrees)___________

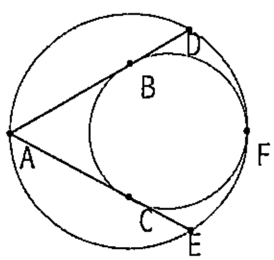
3.) _______ $4\sqrt{10}$ ___________

1). A circle of radius 9 has four points B,C,D,E that are on the circle. BC and DE intersect at point F in the interior of the circle. EF=7 and DE=15. If BC is a diameter of the circle, what two numbers give the lengths of BF and CF?



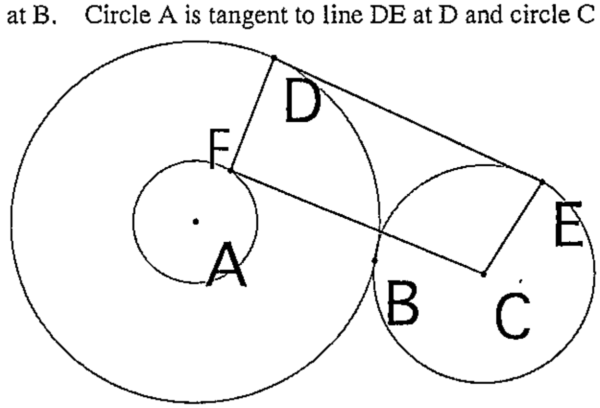
Solution: DF*FE = CF*BF but CF=18-BF and DF=15-EF, so DF=7, so if x is the length of BF, we have 8*7 = x(18-x), so x^2-18x+56=0, so (x-14)(x-4)=0. The lengths are 14 and 4.

2) Two circles are internally tangent at point F. The radius of the smaller circle is $\frac{2}{3}$ of the radius of the larger circle. Point A is on the larger circle. One segment drawn from point A is tangent to the smaller circle at B and intersects the larger circle at D. Another segment drawn from point A is tangent to the smaller circle at C and intersects the larger circle at E. Find the sum of the degree measures of minor arc BC and minor arc DE.



Solution: Suppose the small circle has center O. Then ABO is a right triangle with AO as the hypotenuse and AO = 2 BO, so angle BAO is 30 degreees,. By symmetry, angle OAC is also 30 degrees, so angle BAC is 60 degrees. The measure of arc BC must be 120 degrees because 60 = (240-120)/2. Angle DAE is an inscribed angle for the larger circle, so the measure of arc DE must be 2*60 degrees = 120 degrees.
The sum of the two arcs is 240 degrees.

3) A circle with center A has radius 8 cm. A circle with center C has radius 5 cm. The two circles are externally tangent to one another at B. Circle A is tangent to line DE at D and circle C is tangent to line DE at E. Find the length of DE



Solution: Draw a circle of radius 3 with center A. Suppose AD meets the new circle at F. The length of FC is the same as the length of DE and since F lies on AD, ΔAFC is a right triangle. $(AF)^2+(AC)^2 = (FC)^2$, so $3^2 + (FC)^2 = 13^2$, so FC = DE = $\sqrt{160} = 4\sqrt{10}$

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Match 5 Round 4
Quadratic Equations and Complex Numbers

1.) ______ x^2-26x+180=0______

2.) ____________2+5i, 2-3i______

3.) ____________3-i______________

1) A quadratic equation with integer coefficients has $13-i\sqrt{11}$ as one of its zeros. Find the quadratic equation expressed in the form ax^2 + bx+c =0 with a>0 and a,b,and c relatively prime integers

Since the quadratic has integer coefficients, the other zero must be $13+i\sqrt{11}$. The sum of these numbers is 26 and their product is 169+11 = 180. Since the sum of the roots is –b/a and the product is c/a, the equation is x^2-26x+180=0

2) Find all complex solutions of $x^2-(4+2i)x+4i=-19$.

A=1, B=-4-2i, C=19+4i

$$x=\frac{4+2i\pm\sqrt{(4+2i)^2-76-16i}}{2}$$

$$x=\frac{4+2i\pm\sqrt{12+16i-76-16i}}{2}$$

$$x=\frac{4+2i\pm 8i}{2}=2+5i,2-3i$$

3) Express the following as a complex number a+bi for real numbers a and b:

$$\frac{4i}{(1+i)^3}-\frac{2}{(1-i)^2}+\frac{3+4i}{2+i}$$

$(1+i)^3=(1+i)(1+i)(1+i)=2i(1+i)=-2+2i$ so this is $\frac{4i}{-2+2i}-\frac{2}{-2i}+\frac{3+4i}{2+i}=$

$(1-i)^2=-2i$

$$\frac{2i}{-1+i}+\frac{1}{i}+\frac{(3+4i)(2-i)}{(2+i)(2-i)}==\frac{2i(-1-i)}{(-1+i)(-1-i)}-i+\frac{(3+4i)(2-i)}{(2+i)(2-i)}=1-i-i+2+i=3-i$$

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Match 5 Round 5
Solving Trig Equations

1.) $0, \pi$

2.) $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

3.) ~~$\frac{6 \pm \sqrt{30}}{16}$~~ $\frac{3 \pm \sqrt{7}}{8}$

1) Solve for all x if $0 \le x < 2\pi$: $\sec^3 x - \sec x - \tan^2 x = 0$
$\sec^3 x + \sec x - (\sec^2 x - 1) = 0$, so $\sec^3 x - \sec^2 x - \sec x + 1 = 0$,
so $\sec^2 x (\sec x - 1) - 1 (\sec x - 1) = 0$, so $(\sec^2 x - 1)(\sec x - 1)$, so
$\sec x = 1$ or -1, so $x = 0$ or π

2) Solve for all x if $0 \le x < 2\pi$: $\sin(2x) - \cos(x) = \cot(x)$
$2 \sin x \cos x - \cos x = \frac{\cos x}{\sin x} \frac{\cos x}{\sin x}$, so $2 \sin^2 x \cos x - \sin x \cos x = \cos x$,
so $\cos x (2 \sin^2 x - \sin x - 1) = 0$, so $\cos x (2 \sin x + 1)(\sin x - 1) = 0$,
so $\cos x = 0$ or $\sin x = -1/2$ or $\sin x = 1$, so
$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

3) If $\sin(A+B) - \sin(A-B) = \frac{-1}{16}$ and $\cos A = \sin B + \frac{3}{4}$, find all possible values of $\cos A$. Express your answers in simplest radical form.
$\sin A \cos B + \sin B \cos A - [(\sin A \cos B) - \sin B \cos A] = -1/16$, so
$2 \sin B \cos A = -1/16$.
Substitute $\sin B + ¾$ for $\cos A$, to get $2 \sin B(\sin B + ¾) = -1/16$, so
$2 \sin^2 B + \frac{3}{2} \sin B + \frac{1}{16} = 0$, so $32 \sin^2 B + 24 \sin B + 1 = 0$

$$\sin B = \frac{-24 \pm \sqrt{576 - 4*32}}{64} = \frac{-24 \pm \sqrt{480}}{64} = \frac{-24 \pm 4\sqrt{30}}{64} = \frac{-6 \pm \sqrt{30}}{16}$$

$$\cos A = \frac{-6 \pm \sqrt{30}}{16} + \frac{3}{4} = \frac{-6 \pm \sqrt{30}}{16} + \frac{12}{16} = \frac{6 \pm \sqrt{30}}{16}$$

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Match 5 Round 6 Sequences and Series

1.) 0.996

2.) $\pm\frac{\sqrt[4]{2}}{8}$

3.) 15

1) An infinite geometric series converges to 2500. A second infinite geometric series has the same common ratio, but the first term of the second series is 6 greater than the first term of the first series. The second series converges to 4000. What is the common ratio for each series?

$\frac{a}{1-r} = 2500$ and $\frac{a+6}{1-r} = 4000$, so $\frac{a}{2500} = \frac{a+6}{4000}$, so 4000a=2500(a+6), so 1500a=15000 so a=10. If $\frac{10}{1-r} = 2500$ then 1-r = $\frac{10}{2500}$ = .004, so r= 0.996

2) In a geometric sequence, the fifth term is 16 and the ninth term is 2. If all terms are real numbers, give all possible values for the fourteenth term. Express your answer in simplest radical form.

$a_1 r^4 = 16$ and $a_1 r^8 = 2$, so $r^4 = \frac{1}{8}$ so $r = \pm\sqrt[4]{\frac{1}{8}}$. The first term is 128 in either case. The fourteenth term is $2^7 * ((\pm\sqrt[4]{\frac{1}{8}})^{13}$ = $2^7 * \frac{1}{2^9} * (\pm\sqrt[4]{\frac{1}{8}})$ = $\pm\frac{1}{4}\frac{\sqrt[4]{2}}{\sqrt[4]{16}}$ = $\pm\frac{\sqrt[4]{2}}{8}$

3) The fourth term of an arithmetic sequence is 9.2. The seventh term of the sequence is 16.7. The sum of the first n terms is 288. Find n.

The common difference is (16.7-9.2)/3 = 2.5. The first term is 9.2-3*2.5 = 1.7. We have $288 = \frac{n(2*1.7+(n-1)*2.5)}{2}$ or $288 = \frac{n(2.5n+0.9)}{2}$, so $576 = 2.5n^2 + 0.9n$.
So $25n^2 + 9n - 5760 = 0$. N must be a whole number, so it must factor to (n-?)(25n+?)
(n-15)(25n+384) works, so n=15.

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Match 5 Team Round

1.) $\dfrac{4b^9c^{10}}{a^6}$ 4.) 7-4i, -7+4i

2.) $1+\sqrt{2}$ 5.) $\dfrac{3\pm4\sqrt{3}}{13}$

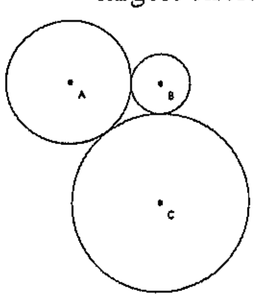
3.) 1:4:9 6.) $\dfrac{-32}{3}$, 1

1) If $m=a^3b^2c^{-5}$ and $n = a^2b^{-3}c^4$ and no variables are equal to zero, simplify $\dfrac{7c^{22}m^2-3b^{13}n^3}{m^2n^3}$. Express your answer with no negative exponents.

$$\frac{7c^{22}(a^3b^2c^{-5})^2-3b^{13}\left(a^2b^{-3}c^4\right)^3}{((a^3b^2c^{-5})^2(a^2b^{-3}c^4)^3}=\frac{7c^{22}(a^6b^4c^{-10})-3b^{13}\left(a^6b^{-9}c^{12}\right)}{(a^6b^4c^{-10})(a^6b^{-9}c^{12})}=$$
$$\frac{7(a^6b^4c^{12})-3\left(a^6b^4c^{12}\right)}{(a^{12}b^{-5}c^2)}=\frac{4(a^6b^4c^{12})}{(a^{12}b^{-5}c^2)}==\frac{4b^9c^{10}}{a^6}$$

2) If x is equal to the infinite continued fraction $2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}$, find the value of x in simplest radical form. Solution: Replace the expression beneath the first 1 by x, so that you get $x = 2+\dfrac{1}{x}$, so $X^2 - 2x - 1 = 0$, so $x = \dfrac{2\pm\sqrt{8}}{2}=\dfrac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$, but since x is clearly positive, choose $1+\sqrt{2}$

3) Each of three circles is tangent externally to the other two circles. The triangle formed by connecting the centers of the three circles is the smallest right triangle such that the radii of all the circles are integers. If x is the area of the smallest circle, y is the area of the middle-sized circle, and z is the area of the largest circle, give the ratio x:y:z.



Solution: Let D be the point where circles A and B meet, E is the point where circles A and C meet, and F be the point where circles B and C meet. The smallest possible right triangle where the sides are integers is a 3-4-5 triangle, so x+y=3, x+z=4, and x+z=5. Solve the system to get x=1, y=2, z=3, and the ratio of the areas is 1:4:9

4) Find the two complex square roots of $33 - 56i$

Let $\sqrt{33-56i} = a+bi$. Square both sides to get $a^2 - b^2 = 33$ and $2abi = -56i$, so $a = -\frac{28}{b}$. Substitute for a to get $(-\frac{28}{b})^2 - b^2 = 33$, so $784 - b^4 = 33b^2$. Factor $b^4 + 33b^2 - 784$ to $(b^2-16)(b^2+49)$, so $b = \pm 4$ or $b = \pm 7i$, but since b must be real, b=4 or -4. If b=4, a=-28/4 = -7. If b=-4, a=-28/-4 = 7, so 7-4i, -7+4i

5) If $3 \cos x + 2 \sin x = 1$, find all possible values of cos x. Express your answers in simplest radical form.

$2 \sin x = 1 - 3 \cos x$. Square both sides to get $4 \sin^2 x = 1 - 6 \cos x + 9 \cos^2 x$, so $4(1-\cos^2 x) = 1 - 6 \cos x + 9 \cos^2 x$, so $13 \cos^2 x - 6 \cos x - 3 = 0$, so

$$x = \frac{6 \pm \sqrt{36-(-156)}}{26} = \frac{6 \pm \sqrt{192}}{26} = \frac{6 \pm 8\sqrt{3}}{26} = \frac{3 \pm 4\sqrt{3}}{13}$$

6) In an arithmetic sequence, the 17^{th} term is the square of the 5th term. If the 11^{th} term is 6, what are all possible values for the first term of the sequence?

The 5^{th} term is 6-6d, and the 17^{th} term is 6+6d, so $(6-6d)^2 = 6+6d$, so $36 - 72d + 36d^2 = 6+6d$, so $36d^2 - 78d + 30 = 0$, so $6d^2 - 13d + 5 = 0$, so $(3d-5)(2d-1) = 0$, so $d = \frac{5}{3} or \frac{1}{2}$. The first term is 6 -10d, so either $6-10(\frac{5}{3})$ or $6-10(\frac{1}{2}) = 6 - \frac{50}{3}$ or 6-5, so the first term is or $\frac{-32}{3}$ or 1.