FAIRFIELD COUNTY MATH LEAGUE 2018-2019

Match 3 Round 1 Arithmetic: Scientific & Base Notation

1.) 5

2.) 4

3.) 126

1.) If $\frac{(3*10^4)^3(8*10^{-3})}{(6*10^5)^2} = a * 10^b$, where $1 \leq a < 10$, find the value of $a + b$.

$\frac{3^3*10^{12}*8*10^{-3}}{6^2*10^{10}} = \frac{6}{10}$ → $a = 6$ and $b = -1$, so $a + b = 5$.

2.) How many permutations of 0's and 1's, up to three digits, have the property that the base 10 value of its base 3 representation is an integer multiple of the base 10 value of its base 2 representation? (For example, 010 would not, since 10_3 in base 10 is not an integer multiple of 10_2 in base 10.)

There are 8 total permutations of 3 digits that can only be 0 or 1. Note that 000 and 001 work since they equal 0 and 1 in both bases respectively.
Checking the remaining 5 (since we know 010 fails)....
$011_3 = 4$ $011_2 = 3$ fails,
$100_3 = 9$ $100_2 = 4$ fails,
$101_3 = 10$ $101_2 = 5$ works,
$110_3 = 12$ $110_2 = 6$ works,
$111_3 = 13$ $111_2 = 7$ fails, so there are 4 permutations total.

3.) If, for positive integers a and b, $123_a - 123_b = 260$ and $a + b = 24$, find the value of bb_a and express it as a base 10 numeral.

Turning the equation into polynomials in terms of a and b yields
$a^2 + 2a + 3 - (b^2 + 2b + 3) = 260 \rightarrow$
$a^2 - b^2 + 2(a - b) = 260 \rightarrow (a - b)(a + b) + 2(a - b) = 260 \rightarrow$
$(a - b)(a + b + 2) = 260.$

Substituting $a + b = 24$ yields $(a - b)(26) = 260 \rightarrow a - b = 10$. You can now solve for a and b, finding $a = 17$ and $b = 7$. Calculating 77_{17} in base 10 yields $7(17) + 7 = 126$.

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Match 3 Round 2
Algebra 1: Word Problems

1.) 2.4

2.) $\frac{28}{9}$ hours

3.) 25 liters

1.) If at a health food store cashews sell for \$12 per pound and walnuts sell for \$5 per pound, how many pounds of cashews are in a 5-pound mix that sells for \$41.80?

Let c represent the number of pounds of cashews. Setting up the equation $12c + 5(5 - c) = 41.80$ and solving yields $7c = 16.80 \rightarrow c = 2.4$.

2.) Minnie Vann drives half way to Paytoo Park at an average speed of r miles per hour. She drives the second half at an average speed of $2r$ miles per hour. If it took 7 hours total for her to drive to Paytoo Park, how many hours would it have taken her to drive the entire way at an average speed of $3r$ miles per hour?

Let D represents the total distance to Paytoo Park. Knowing the time for each leg is the distance divided by the average speed yields the equation $\frac{\frac{D}{2}}{r} + \frac{\frac{D}{2}}{2r} = 7 \rightarrow \frac{D}{2r} + \frac{D}{4r} = 7 \rightarrow \frac{3D}{4r} = 7 \rightarrow \frac{D}{r} = \frac{28}{3}$. Dividing both sides by 3 yields $\frac{D}{3r} = \frac{28}{9}$, so the total time at a speed of $3r$ would be $\frac{28}{9}$ hours.

3.) Bucket A and Bucket B contain water. After five liters are poured from bucket A to Bucket B without spilling, Bucket B contains three times as much water as bucket A. If the original amount of water in bucket A is *then* poured from bucket B to bucket A without spilling, bucket B will contain 60% of the amount of water in bucket A. How many liters of water were originally in bucket B?

Let A represent the original amount of water in bucket A, and let B represent the original amount of water in bucket B. The first statement yields the equation $3(A-5)=B+5$. The second statement yields the equation $\frac{3}{5}(A-5+A)=B+5-A$. Simplifying both equations and solving for B yields the system $\begin{cases} 3A-20=B \\ \frac{11}{5}A-8=B \end{cases}$. Subtracting the equations yields $\frac{4}{5}A-12=0$, so $A=15$ liters and $B=25$ liters.

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Match 3 Round 3 Geometry: Polygons

1.) 172

2.) 9

3.) 7, 18

1.) If a regular octagon has exterior angles measuring d degrees, what is the measure of one interior angle of a regular d-gon in degrees?

The measure of one exterior angle of a regular octagon is $\frac{360}{8} = 45$ degrees, so the measure of one interior angle of a regular 45-gon is $180 - \frac{360}{45} = 172$ degrees.

2.) A particular n-gon has the property that multiplying the number of sides by one-third the number of sides gives the exact number of diagonals in the n-gon. Find the value of n.

Turning the relationship into an equation yields $n\left(\frac{1}{3}n\right) = \frac{n^2-3n}{2} \rightarrow \frac{1}{3}n^2 = \frac{1}{2}n^2 - \frac{3}{2}n \rightarrow n^2 - 9n = 0 \rightarrow n(n-9) = 0 \rightarrow n = 9$.

3.) A particular regular k-gon has the property that ten times the total degree measure of all of its interior angles is exactly equal to the sum of twenty times the total degree measure of its exterior angles and the product of the number of its diagonals and the degree measure of one of its interior angles. Find all possible values of k.

Setting up the equation:

$$10(180k - 360) = 20(360) + \left(\frac{k^2 - 3k}{2}\right)\left(180 - \frac{360}{k}\right)$$

$$1800k - 3600 = 7200 + (k^2 - 3k)\left(90 - \frac{180}{k}\right)$$

$$1800k - 3600 = 7200 + 90k^2 - 180k - 270k + 540$$

$$90k^2 - 2250k + 11340 = 0$$

$$k^2 - 25k + 126 = 0$$

$$(k - 18)(k - 7) = 0$$

$k = 18$ or $k = 7$

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Match 3 Round 4
Algebra 2: Functions & Inverses

1.) $y = \frac{9}{4}x - 15$

2.) $-\frac{2}{3}, \frac{\sqrt{10}}{2}, -\frac{\sqrt{10}}{2}$

3.) Domain: $[2, \infty)$ Range: $\left(0, \frac{1}{16}\right]$

Note: the inverse f^{-1} of a function f is not necessarily a function.

1.) If $f(x) = \frac{2}{3}x + 4$, find the equation for the graph of $f^{-1}(f^{-1}(x))$ in slope-intercept ($y = mx + b$) form.

Setting up $x = \frac{2}{3}y + 4$ yields the inverse $f^{-1}(x) = \frac{3}{2}x - 6$, and $f^{-1}(f^{-1}(x)) = \frac{3}{2}\left(\frac{3}{2}x - 6\right) - 6 = \frac{9}{4}x - 15$.

2.) If $h(x) = \frac{3x^2+10x+10}{(3x+5)(2x-1)}$ and (a, b) is a point on the graphs of both h and h^{-1}, find all possible values of a.

Since any points on both the graphs of h and h^{-1} must have the property that $y = x$, it follows that we can solve for the x-values by setting up $x = \frac{3x^2+10x+10}{(3x+5)(2x-1)} \rightarrow x(3x + 5)(2x - 1) = 3x^2 + 10x + 10 \rightarrow$
$6x^3 + 7x^2 - 5x = 3x^2 + 10x + 10 \rightarrow 6x^3 + 4x^2 - 15x - 10 = 0 \rightarrow$
$2x^2(3x + 2) - 5(3x + 2) = 0 \rightarrow (2x^2 - 5)(3x + 2) = 0 \rightarrow$
$x = -\frac{2}{3}, \pm\frac{\sqrt{10}}{2}$.

3.) If $f(3x+1) = 4\sqrt{x-5} + 2$ and $g(x) = \frac{1}{f^{-1}(x)}$, find the domain and range of $g(x)$. If you use inequalities to represent the domain and range, use x for the domain and y for the range.

If $f(3u+1) = 4\sqrt{u-5} + 2$, let $x = 3u + 1$ so $u = \frac{x-1}{3}$. So $f(x) = 4\sqrt{\frac{x-1}{3} - 5} + 2 = 4\sqrt{\frac{x-16}{3}} + 2$. Therefore the domain of f (and the range of f^{-1}) is $[16, \infty)$ and the range of f (and the domain of f^{-1}) is $[2, \infty)$. The domain of $g(x)$ will be the same as that of $f^{-1}(x)$ since $f^{-1}(x)$ is never equal to zero. The maximum value of the range of $g(x)$ is the reciprocal of the minimum value of the range of $f^{-1}(x)$, and from there the range of $g(x)$ decreases asymptotically to 0, so the range of $g(x)$ is $\left(0, \frac{1}{16}\right]$.

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Match 3 Round 5 Precalculus: Exponents & Logarithms

1.) 400

2.) $\frac{3}{8}$

3.) $125, \sqrt[4]{5}$

1.) If $a^x = 4$ and $\log_a 5 = y$, find a^{2x+2y}.

Since $a^y = 5$, it follows that $a^{2x+2y} = (a^{x+y})^2 = (a^x a^y)^2 = (4 * 5)^2 = 400$.

2.) If $36^{2\log_6 b} = 27^{\log_9 a}$, find $\log_a(b)$.

Since $36^{2\log_6 b} = (6^2)^{2\log_6 b} = 6^{4\log_6 b} = b^4$ and $27^{\log_9 a} = \left(9^{\frac{3}{2}}\right)^{\log_9 a} = a^{\frac{3}{2}}$, it follows that $b^4 = a^{\frac{3}{2}} \rightarrow b = a^{\frac{3}{8}}$, so $\log_a(b) = \frac{3}{8}$.

3.) Solve for all real values of x: $\log_5(x^4) + \log_x\left(\frac{125}{x}\right) = 12$

$$\log_5(x^4) + \log_x\left(\frac{125}{x}\right) = 12$$
$$4\log_5 x + \log_x 125 - \log_x x = 12$$
$$4\log_5 x + \frac{\log_5 125}{\log_5 x} - 1 = 12$$

$$4\log_5 x + \frac{3}{\log_5 x} - 13 = 0$$

$$4(\log_5 x)^2 - 13\log_5 x + 3 = 0$$

Letting $u = \log_5 x$, we get the equation $4u^2 - 13u + 3 = 0 \rightarrow$

$(u-3)(4u-1) = 0 \rightarrow u = 3, \frac{1}{4} \rightarrow \log_5 x = 3, \frac{1}{4} \rightarrow x = 125, \sqrt[4]{5}$

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Match 3 Round 6
Miscellaneous: Matrices

1.) $\frac{21}{2}$

2.) -12

3.) $\frac{1\pm\sqrt{43}}{2}, \frac{1\pm\sqrt{47}}{2}$

1.) If $\begin{bmatrix} 3 & b \\ b & c \end{bmatrix} + 2\begin{bmatrix} a & a \\ c & d \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ 6 & 20 \end{bmatrix}$, find the value of d.

Note that $3 + 2a = 13$, so $a = 5$. Then note $b + 2a = 18 \rightarrow b + 2(5) = 18 \rightarrow b = 8$. Then $b + 2c = 6$, so $c = -1$. Finally $c + 2d = 20$, so $d = \frac{21}{2}$.

2.) If $B = \begin{bmatrix} 5 & 1 \\ 8 & 2 \end{bmatrix}$ and $AB = \begin{bmatrix} 7 & 1 \\ 3 & -3 \end{bmatrix}$, find the determinant of A.

One way to do this problem is to compute A manually by finding B^{-1} and then computing $(AB)B^{-1}$. $B^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -8 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -4 & \frac{5}{2} \end{bmatrix}$, and $A = (AB)B^{-1} = \begin{bmatrix} 3 & -1 \\ 15 & -9 \end{bmatrix}$, so the determinant of $A = -12$. Another way is to invoke the property that $\det(A)\det(B) = \det(AB)$. Since $\det(B) = 2$ and $\det(AB) = -24$, it follows that $\det(A) = -12$.

3.) If C is a non-singular (invertible) matrix such that $\det(C) = \det(C^{-1})$, and $C = \begin{bmatrix} k & 2 & k+3 \\ k & 3 & -1 \\ 2 & 1 & k \end{bmatrix}$, find all possible values of k.

Knowing that $\det(C) = \frac{1}{\det(C^{-1})}$, we can conclude that $\det(C) = \pm 1$. Calculating the determinant yields the equation $k(3k+1) - 2(k^2+2) + (k+3)(k-6) = \pm 1$, which simplifies to $2k^2 - 2k - 22 = \pm 1$. In the first case, $2k^2 - 2k - 23 = 0$, so $k = \frac{2 \pm \sqrt{188}}{4} = \frac{1 \pm \sqrt{47}}{2}$. In the second case, $2k^2 - 2k - 21 = 0$, so $k = \frac{2 \pm \sqrt{172}}{4} = \frac{1 \pm \sqrt{43}}{2}$.

Team Round

FAIRFIELD COUNTY MATH LEAGUE 2018-19 Match 3 Team Round

1.) D: $[1,10) \cup (10,\infty)$; R: $\left(0,\frac{1}{6}\right) \cup \left(\frac{1}{6},\frac{1}{3}\right]$

2.) 9

3.) $\pm\frac{\sqrt{47}}{2}, \pm\sqrt{5}$

4.) -2

5.) 17

6.) $\left(\frac{200}{11},\frac{200}{9}\right)$

1.) Find the domain and range of $f(x) = \frac{\sqrt{x-1}-3}{x-10}$. If you use inequalities, use x for the domain and y for the range.

Note that the domain excludes $x = 10$ and otherwise includes all values $x \geq 1$, so the domain is $[1,10) \cup (10,\infty)$. Also note that $x = 10$ is indeterminate. Multiplying the numerator and denominator by the conjugate of the numerator yields $\frac{1}{\sqrt{x-1}+3}$, and letting $x = 10$ shows that this point is a removable discontinuity with a y-value of $\frac{1}{6}$. Finally, noting the denominator has a minimum value of 3 and is not bounded above yields a range of $\left(0,\frac{1}{6}\right) \cup \left(\frac{1}{6},\frac{1}{3}\right]$.

2.) Find all possible values of x if $1001_{x+3} + 121_x = 725_{2x-2}$.

Translating the statement into an equation in terms of x yields:
$(x+3)^3 + 1 + x^2 + 2x + 1 = 7(2x-2)^2 + 2(2x-2) + 5$
$x^3 + 9x^2 + 27x + 27 + 1 + x^2 + 2x + 1 = 7(4x^2 - 8x + 4) + 4x - 4 + 5$
$x^3 - 18x^2 + 81x = 0 \rightarrow x(x-9)^2 = 0 \rightarrow x = 9$.

3.) A and B are matrices such that $AB = BA$. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 2p^2+3p & p^2+q^2 \end{bmatrix}$, find all possible values of q.

Setting $AB = BA$ yields an equation for the top left (first row, first column) element: $5 + 2(2p^2+3p) = 5 + 18 \rightarrow 4p^2 + 6p - 18 = 0 \rightarrow 2p^2 + 3p - 9 = 0 \rightarrow$ $(2p-3)(p+3) = 0 \rightarrow p = \frac{3}{2}, -3$. Setting the two matrix products equal also yields the following equation for the top right (first row, second column) element of the product: $6 + 2(p^2+q^2) = 10 + 24 \rightarrow p^2 + q^2 = 14$. Letting $p = \frac{3}{2}$ yields $q = \pm\frac{\sqrt{47}}{2}$, and letting $p = -3$ yields $q = \pm\sqrt{5}$.

4.) Find all values of t such that $1 + \log_9(t+6) = \log_3(4-t)$.

Note that $\log_3(4-t) = 2\log_9(4-t) = \log_9(4-t)^2$. Therefore, rewriting the equation: $1 + \log_9(t+6) = \log_9(4-t)^2 \rightarrow 1 = \log_9(4-t)^2 - \log_9(t+6) \rightarrow$ $1 = \log_9\left(\frac{(4-t)^2}{t+6}\right) \rightarrow 9 = \frac{(4-t)^2}{t+6} \rightarrow 9(t+6) = (4-t)^2 \rightarrow t^2 - 17t - 38 = 0 \rightarrow$ $(t-19)(t+2) = 0 \rightarrow t = -2, 19$. Note that $t = 19$ is an extraneous solution, so the only solution is $t = -2$.

5.) The measure of one interior angle of regular a k-gon is the same as the measure of one exterior angle of a regular m-gon. Find the sum of all possible distinct values of $k + m$.

Intuitively, we could notice that in order for an exterior angle to be equal to the measure of an interior angle, the exterior angle cannot have a measure less than 60 degrees, which means $m \leq 6$. From here we can check individual cases.

Alternatively, we could set up $180 - \frac{360}{k} = \frac{360}{m} \rightarrow 1 - \frac{2}{k} = \frac{2}{m} \rightarrow k = \frac{2m}{m-2} = 2 + \frac{4}{m-2}$, so this will only work when $\frac{4}{m-2}$ is an integer for integer values of $m \geq 3$. For ordered pairs (k, m), this yields possibilities of (3,6), (4,4), and (6,3). Since we only want distinct values of $k + m$, this means the sum is $9 + 8 = 17$.

6.) For nonzero numbers x and y, twenty times the average (arithmetic mean) of x and y is equal to the product of x and y. If y is the same as x increased by $y\%$, find the ordered pair (x, y).

Creating the two equations as written yields $\begin{cases} 20\left(\frac{x+y}{2}\right) = xy \\ y = x\left(1 + \frac{y}{100}\right) \end{cases}$. Isolating xy in each equation yields the system: $\begin{cases} 10x + 10y = xy \\ 100y - 100x = xy \end{cases}$. Subtracting the equations gives us $10x + 10y - 100y + 100x = 0 \rightarrow y = \frac{11}{9}x$. Substituting back into the first equation gives $10x + 10\left(\frac{11}{9}x\right) = x\left(\frac{11}{9}x\right) \rightarrow \frac{200}{9}x = \frac{11}{9}x^2 \rightarrow x = \frac{200}{11}$, making $y = \frac{200}{9}$.