FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 4 Round 1
Arithmetic:
Basic Statistics

1.) ___________6.4____________

2.) __________14706___________

3.) __________285_______________

1.)The geometric mean of the numbers x_1, x_2, … x_n is defined to be $\sqrt[n]{x_1 x_2 \ldots x_n}$. What is the positive difference between the arithmetic mean and the geometric mean of the set of numbers

{1, 25, 1, 5, 25} ?

The geometric mean is $\frac{1+25+1+5+25}{5} = \frac{57}{5} = 11.4$ $\sqrt[5]{1*5^2*1*5*5^2} = \sqrt[5]{5^5} = 5$. The

arithmetic mean is

$\frac{1+25+1+5+25}{5} = \frac{57}{5} = 11.4$. 11.4-5 = 6.4

2.) The upper quartile is defined to be the median of the upper half of a set of numbers. The lower quartile is defined to be the median of the lower half of a set of numbers. What is the product of the arithmetic mean, median, upper quartile, and lower quartile of the set consisting of the ten smallest prime numbers?

The ten smallest prime numbers are 2,3,5,7,11,13,17,19,23,29. The median is halfway between 11 and 13, which is 12. The mean is (2+3+5+7+11+13+17+19+23+29)/10 = 12.9. The median of the lower half is 5, and the median of the upper half is 19, so the required product is 12*5*19*12.9=60*19*12.9=14706

3.) A set of 10 numbers has a certain arithmetic mean. If you remove two numbers that add to 105, the arithmetic mean of the remaining numbers is 6 less than the arithmetic mean of the original ten numbers. What is the sum of the original 10 numbers?

Let x=mean of original 10 numbers, and y=sum of original 10 numbers.

Then 10x=y. Also 8(x-6)=y-105, so 8(x-6)=10x-105, so 8x-48=10x-105, so 2x=57, and x=28.5. The sum of the original 10 numbers is 10x, so 10*28.5 = 285.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 4 Round 2
Algebra 1:
Quadratic
Equations

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1.) ________ $8, \frac{-4}{7}$ __________

2.) ___________ $3, \frac{1}{k}$ __________

3.) ________2,5,6,7____________

1.) Find the two solutions of the equation $9(x+2)^2 = (4x-2)^2$.

$9(x^2+4x+4)=(16x^2-16x+4)$

$9x^2+36x+36=16x^2-16x+4$

$7x^2-52x-32=0$

$(7x+4)(x-8) = 0$ $x=8$ or $x= \frac{-4}{7}$

2.) If $k\neq 0$, find all values of x in terms of k such that $kx^2 + 3 = (3k+1)x$.

This can either be solved by factoring $kx^2-3kx-3x+3=0$ to

$Kx(x-3)-1(x-3)=0$, so $(kx-1)(x-3)=0$, so $x=3$ or $x= \frac{1}{k}$, or by quadratic formula

$$\frac{(3k+1)\pm\sqrt{(3k+1)^2 - 4*k*3}}{2k}$$

$$= \frac{(3k+1)\pm\sqrt{(9k^2 - 6k+1)}}{2k}$$

$$= \frac{(3k+1)\pm|3k-1|}{2k} = \frac{3k+1+3k-1}{2k} = \frac{6k}{2k} = 3$$

$$or \frac{3k+1+1-3k}{2k} = \frac{2}{2k} = 3, \frac{1}{k}$$

3.) Find all positive integer values of m such that $m^2x^2 - 52x + \frac{96}{m} = 0$ has two rational solutions.

Check the discriminant $(-52)^2-4m*96$ for all possible values of m and see which ones give perfect squares.

If m=1, 2704-384 = 2320, not a perfect square

If m=2, $2704-768=1936=44^2$. OK! If m=3, 2704-1152=1552, not a perfect square. If m=4, 2704-1536=768, not a perfect square. If m=5, $2704-1920=784=28^2$ Ok! If m=6, $2704-2304=400=20^2$. OK! If m=7, $2704-2688=16=4^2$. Ok! If m>7, the discriminant is negative.

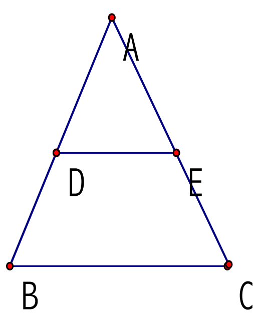
FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 4 Round 3
Geometry:
Similarity

1.) ________2.2______________

2.) _________3:2_______________

3.) ________2.4 cm______________



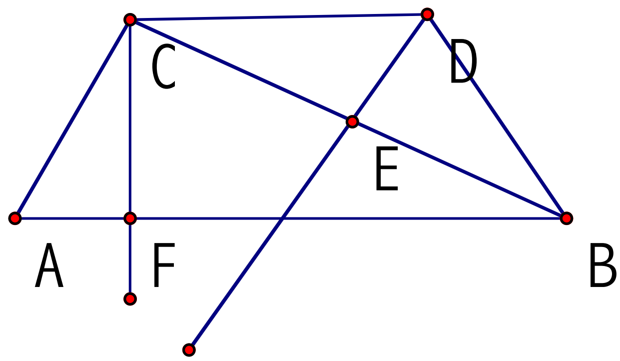
1) In the figure above, $\overline{BC} \parallel \overline{DE}$. If AC=12, BC=10, DE=8, and AB=11, find the length of BD.

ΔADE is similar to ΔABC, so $\frac{AD}{AB} = \frac{DE}{BC}$. If x is the length of AD, then

$\frac{11-x}{11} = \frac{8}{10}$, so 10(11-x)=88, so 110-10x=88, so -10x=-22, so x=2.2

2.) Two regular hexagons are such that the larger hexagon has area $\frac{243\sqrt{3}}{2}$ cm^2 and the perimeter of the smaller hexagon is 36 cm. What is the ratio of the lengths of the sides of the hexagons? (give two relatively prime integers; give (side of larger hexagon):(side of smaller hexagon)

One side of the smaller hexagon must be 6 cm. The area of each of the 6 triangles making the smaller hexagon is $\frac{6^2\sqrt{3}}{4} = 9\sqrt{3}$, so the total area of the smaller hexagon must be $54\sqrt{3}$. The ratio of the lengths is the same as the ratio of the square root of their areas, so the ratio is $\sqrt{\frac{\frac{243\sqrt{3}}{2}}{54\sqrt{3}}} = \sqrt{\frac{243}{108}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$



3.) Isosceles trapezoid ABCD has area $16\sqrt{3}$ cm^2, AC=4 cm, and the measure of angle CAB is 60 degrees. The bisector of angle CDB intersects diagonal CB at E. Find the length of DE.

Since AC=4 and the angle CAB is 60 degrees, the altitude from C to AB meeting AB at F must be $2\sqrt{3}$ cm, and the length of AF must be 2 cm. Since the trapezoid is isosceles, there is a triangle congruent to ACF involving BD on the other side. The area of this triangle is $(1/2)(2)(2\sqrt{3}) = 2\sqrt{3}$, so the two triangles make up an area of $4\sqrt{3}$, and the rectangle is the center must have area $12\sqrt{3}$, so CD=6, and AB=10. Angle CDB must measure 120 degrees, and since DE bisects angle CDB, angle CDE measures 60 degrees. Angle DCE is congruent to angle ABC by alternate interior angles, so ΔABC is similar to ΔDCE, so $\frac{AB}{AC} = \frac{DC}{DE}$, so $\frac{10}{4} = \frac{6}{DE}$, so DE=2.4 cm.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 4 Round 4
Algebra 2:
Variation

1.) ________12______________

2.) _______ $W = \frac{30}{T^3}$ __________

3.) ____________ $\sqrt[6]{2}$ _______

1.) The number of people needed to do a job varies directly with the amount of work to be done and inversely with the time in which the job is to be completed. If 5 people are required to wash 20 cars in 2 hours, how many people are required to wash 192 cars in 8 hours?

N=number of people, W=work done, T=time.

$\left(\frac{5}{2}\right)^n = \frac{3.75}{0.24} = \frac{\frac{30}{8}}{\frac{6}{25}} = \frac{30*25}{8*6} = \frac{750}{48} = \frac{125}{8}$ $N = k\frac{W}{T}, 5 = k\frac{20}{8}$, so k=1/2.

$N = \frac{1}{2}\frac{192}{8}$=12.

2.) W varies inversely with some power of T. When T=2, W=3.75. When T=5, W=0.24. Find the function relating W and T. Express your answer as $W = \frac{k}{T^n}$, for numbers k and n. 3.75 = $W = \frac{k}{T^n}$, and 0.24 = $\frac{k}{5^n}$. So

$\left(\frac{5}{2}\right)^n = \frac{3.75}{0.24} = \frac{\frac{30}{8}}{\frac{6}{25}} = \frac{30*25}{8*6} = \frac{750}{48} = \frac{125}{8}$, so n=3. 3.75 = $\frac{k}{2^3}$, so k=30. The function is

$W = \frac{30}{T^3}$

3.) z varies jointly with the square of x and the cube of y, and y varies inversely with the fourth power of w. If z= $\frac{3}{64}$ when x=6 and w=2, what is w when z=108 and x=9?

$Z=k_1x^2y^3$, and $y= k_2/w^4$. So $z = (k_1k_2)^3x^2/w^{12}$. Call $(k_1k_2)^3=K$, so $z=Kx^2/w^{12}$.

$\frac{3}{64}$ = $K*6^2/2^{12}$, so K= $\frac{3*2^{12}}{64*6^2} = \frac{3*64}{6^2} = \frac{16}{3}$. Then solve 108= $\frac{16}{3}\frac{9^2}{w^{12}}$, so w^{12}= $\frac{16*9^2}{108*3}$ = $\frac{16*81}{108*3}$=4. So w= $\sqrt[12]{4} = \sqrt[6]{2}$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 4 Round 5
Trig Expressions

1.) ________ $\frac{3\sqrt{2}}{4}$ ________________

2.) ____ $\sin^2(x) - \sin^4(x)$ ______

3.) __________ -32 __________________

1.) Express as a single fraction:

$$\sin(\frac{2\pi}{3})\cos(\frac{3\pi}{4})\cot(\frac{5\pi}{6})\tan(\frac{5\pi}{4}) = \frac{\sqrt{3}}{2} * \frac{-\sqrt{2}}{2} * \frac{-\sqrt{3}}{1} * 1 = \frac{3\sqrt{2}}{4}$$

2.) If neither cos(x) nor sin(x) is 0, simplify and express using no other trig functions except sin(x): The middle sign is a division sign, not an addition sign.

$$\frac{\sec(x)+\tan(x)}{\csc^2(x)\tan^2(x)} \div \frac{\sec^2(x)\cot^2(x)}{\sec(x)-\tan(x)} =$$

$$\frac{\sec(x)+\tan(x)}{\csc^2(x)\tan^2(x)} * \frac{\sec(x)-\tan(x)}{\sec^2(x)\cot^2(x)} =$$

$$\sin^2(x)\cos^2(x)(\sec^2(x) - \tan^2(x)) =$$

$$= \sin^2(x)\cos^2(x)(1)$$

$$= \sin^2(x)(1 - \sin^2(x))$$

$$= \sin^2(x) - \sin^4(x)$$

3.) If you express sin(3x)cos(4x) as a polynomial in terms of sin(x), what is the numerical coefficient of the term involving $\sin^7(x)$?

$$\sin(3x)\cos(4x) =$$

$$\sin(2x + x)\cos(2 * 2x) =$$

$$(\cos(2x)\sin(x) + \sin(2x)\cos(x))(2\cos^2(2x) - 1) =$$

$$(\cos^2(x) - \sin^2(x))(\sin(x)) + 2\sin(x)(\cos^2(x))((2\cos^2(2x) - 1)$$

$$= (-\sin^3(x) + 3\sin(x)(\cos^2(x)))((2\cos^2(2x) - 1))$$

$$= (-\sin^3(x) + 3\sin(x)(1 - \sin^2(x)))((2\cos^2(2x) - 1)) =$$

$$(-4\sin^3(x) + 3\sin(x))((2\cos^2(2x) - 1)) =$$

$$(-4\sin^3(x) + 3\sin(x))((2(1 - 2\sin^2(x))(1 - 2\sin^2 x) - 1))$$

The first polynomial has $-4\sin^3(x)$, the second one has an 8 $\sin^4(x)$, so the coefficient of $\sin^7(x)$ will be -32.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 4 Round 6
Conics

1.) ________ $\sqrt{19}$ _______________

2.) ____ $(-2+2\sqrt{3},1)$__________

3.) ____________ $2\sqrt{30}$ ___________

1.) What is the radius of the circle $2x^2 - 10x + 2y^2 - 14y - 1 = 0$?

Complete the square on x and y to get $2(x^2-5x) + 2(y^2-7y) = 1$
$2(x^2-5x+6.25)+2(y^2-7y+12.25)=1+2*6.25+2*12.25=1+12.5+24.5=38$, so it's $2(x-2.5)^2 + 2(y-3.5)^2 = 38$, so $(x-2.5)^2 + (y-3.5)^2 = 19$, so the answer is $\sqrt{19}$.

2.) The major axis of an ellipse is parallel to the x-axis. The length of the major axis is twice the length of the minor axis. The sum of the distances from any point on the ellipse to each focus is 8. If the ellipse passes through the points (-2,3) and (-2,-1), give the coordinates of the focus of the ellipse that is in the first quadrant.
2a=8, so a=4. Since a=2b, b=2. Since the points (-2,-1) and (-2,3) differ by 4 vertically, they must be on the minor axis, so the center of the ellipse is the midpoint of this segment (-2,1). We have $a^2-b^2=c^2$, so $4^2-2^2= c^2=12$, so $c=2\sqrt{3}$. The foci are $2\sqrt{3}$ horizontally away from (-2,1), so they are at
$(-2-2\sqrt{3},1)$ and $(-2+2\sqrt{3},1)$, and the one we want is $(-2+2\sqrt{3},1)$,

3.) A hyperbola has equation $x^2-2x-2y^2+12y=7$. Find the length of the horizontal line segment that passes through one of the foci and has its endpoints on the asymptotes.

Complete the square to get $x^2-2x+1-2(y^2-6y+9)=7+1-18=-10$.

Divide by -10 to get $\frac{(y-3)^2}{5} - \frac{(x-1)^2}{10} = 1$, so the center is at (1,3). Each focus is c units from (1,3) vertically, where $c^2=15$, so $c=\sqrt{15}$ so the foci are at $(1,3+\sqrt{15})$ and $(1,3-\sqrt{15})$. The slopes of the asymptotes are $\pm \frac{\sqrt{5}}{\sqrt{10}} = \pm\frac{\sqrt{2}}{2}$. They pass through the point (1,3), so say we want to get to the horizontal level of $(1,3+\sqrt{15})$ starting at (1,3) – the change in y is $\sqrt{15}$, so the change in x must be $\sqrt{2}*\sqrt{15}=\sqrt{30}$. The length of the horizontal line segment is twice this number, so the length of the line segment is $2\sqrt{30}$.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014
Match 4 Team Round

1.) __________ $\frac{-1+2\sqrt{3}}{12}$ __________

2.) ____________ $128\sqrt[3]{2}$ __________

3.) ____h(x)= $\frac{3\sqrt{2}}{(\cos(x))^2}$ ____________

4.) ________ $\frac{12\sqrt{5}+15}{5}$ __________

5.) ____0, $\pm\frac{\sqrt{7}}{7}$ ______________

6.) _______60, $\frac{575}{12}$ ____________

1.)If x= $\frac{p}{6}$, find the positive difference between the arithmetic mean and the median of the 6 numbers sin(x), cos(2x), cot(3x), tan(4x), csc(5x), and sec(6x).

$\sin(\frac{p}{6})=\frac{1}{2}$, $\cos(\frac{p}{3})=\frac{1}{2}$, $\cot(\frac{p}{2})=0$, $\tan(\frac{4p}{6})=-\sqrt{3}$, $\csc(\frac{5p}{6})=2$, sec(π)=-1

The median is halfway between 0 and ½, so the median is $\frac{1}{4}$. The sum is $2-\sqrt{3}$, so the arithmetic mean is $\frac{2-\sqrt{3}}{6}$. The difference between these is $\frac{3}{12}-\frac{4-2\sqrt{3}}{12}=\frac{-1+2\sqrt{3}}{12}$

2.) f(x) varies directly with the $\frac{3}{2}$ power of x, and g(x) varies inversely with the $\frac{2}{3}$ power of x. f(64)=g(64)=8. What is $f^{-1}(g(8))$?

$8=k_1(64)^{\frac{3}{2}}$, $\frac{8}{(64)^{\frac{3}{2}}}=k_1=\frac{8}{512}=k_1, \frac{1}{64}=k_1$. $8=\frac{k_2}{64^{\frac{2}{3}}}$, so $k_2=8*64^{\frac{2}{3}}$, so $k_2=128$

To find $f^{-1}(x)$, solve x=$\frac{1}{64}y^{\frac{3}{2}}$ for y, to get y=$(64x)^{\frac{2}{3}}=16x^{\frac{2}{3}}$. g(8)= $\frac{128}{8^{\frac{2}{3}}}=\frac{128}{4}$=32.

$f^{-1}(32)=16(32)^{\frac{2}{3}}=16(2^5)^{\frac{2}{3}}=16(2^{\frac{10}{3}})=16(2^3)\sqrt[3]{2}=128\sqrt[3]{2}$

3) h(x) is defined for $0 \le x < \frac{p}{2}$, and h(x) varies inversely with some power of cos(x). When $x=\frac{p}{6}$, h(x) $=4\sqrt{2}$. When $x=\frac{p}{3}$, h(x) = $12\sqrt{2}$. Express h(x) in the form

$h(x)=\dfrac{k}{(\cos(x))^n}$ for the correct values of k and n.

$4\sqrt{2} = \dfrac{k}{(\cos(\frac{p}{6}))^n} = \dfrac{k}{(\frac{\sqrt{3}}{2})^n}$, and $12\sqrt{2} = \dfrac{k}{(\cos(\frac{p}{3}))^n} = \dfrac{k}{(\frac{1}{2})^n}$. Dividing one equation by the other gives $(\sqrt{3})^n = 3$, so n=2. If $12\sqrt{2} = \dfrac{k}{(\frac{1}{2})^2}$, then k= $3\sqrt{2}$, so h(x)= $\dfrac{3\sqrt{2}}{(\cos(x))^2}$

4) Two of the three intersection points of the circle $x^2+y^2=25$ and $y=\frac{1}{2}x^2-5$ have the same y-coordinate. These two points are the foci of a hyperbola for which the slopes of the two asymptotes are 2 and -2. If the equation of the hyperbola is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$, give the sum of the numbers a,b,h, and k given that a>0 and b>0.

The parabola has equation $2y=x^2-10$, so $-x^2+2y=-10$ Add this to the equation of the circle to get
$Y^2+2y-15=0$, so (y+5)(y-3)=0, so y=-5 or y=3. If y=-5, then x=0. If y=3, then x=±4, so the two foci of the hyperbola are (-4,3) and (4,3), so the center of the hyperbola is at (0,3). c=4, and $\frac{b}{a} = \pm 2$, so $b = \pm 2a$. $A^2+b^2=c^2$, so $a^2+4a^2=16$, so $a^2=\frac{16}{5}$, and $a=\pm\frac{4\sqrt{5}}{5}$, so $b=\pm\frac{8\sqrt{5}}{5}$. H=0, k=3, so add $\frac{4\sqrt{5}}{5}+\frac{8\sqrt{5}}{5}+3=\frac{12\sqrt{5}+15}{5}$

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5) Find all values of tan(x) such that tan(3x)=5 tan(x).

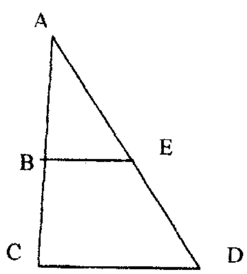
tan(3x)=tan(2x+x)= $\frac{\tan(2x)+\tan(x)}{1-\tan(2x)\tan(x)}$=

$$=\frac{\frac{2\tan(x)}{1-\tan^2(x)}+\tan(x)}{1-\frac{2\tan(x)}{1-\tan^2(x)}\tan(x)}=\frac{\frac{3\tan(x)-\tan^3(x)}{1-\tan^2(x)}}{\frac{1-3\tan^2(x)}{1-\tan^2(x)}}=\frac{3\tan(x)-\tan^3(x)}{1-3\tan^2(x)}$$

. Set this equal to 5*tan(x),

so solve 3*tan(x)-$\tan^3$(x) = 5***tan(x)-15**$\tan^3$(x). One solution is tan(x)=0. If tan(x)≠0, divide through by tan(x) to get 3 – $\tan^2$(x)= 5-15*$\tan^2$(x), so 14*$\tan^2$(x)=2,

so tan(x)= $\pm\frac{\sqrt{7}}{7}$



6.) In the figure above $\overline{BE}$ is parallel to $\overline{CD}$. If BE=10, DE=2x+5, AD=15x, AB=14x-10, and BC=4x. What are the possible values for the perimeter of ΔACD?

$\frac{AB}{AC}=\frac{AE}{ED}$, so, so $\frac{14x-10}{18x-10}=\frac{13x-5}{15x}$, so

15x(14x-10)=(18x-10)(13x-5), so $210x^2$-150x = $234x^2$-220x+50, so $24x^2$-70x+50=0, and

$12x^2$-35x+25=0, so (3x-5)(4x-5)=0, so x=$\frac{5}{3}$ or x=$\frac{5}{4}$. If x=$\frac{5}{3}$, $\frac{AB}{AC}=\frac{BE}{CD}$, so $\frac{\frac{40}{3}}{20}=\frac{10}{CD}$,

so CD=15, and AC+CD+AD=20+15+25=60.

If x=$\frac{5}{4}$, $\frac{AB}{AC}=\frac{BE}{CD}$, so $\frac{7.5}{12.5}=\frac{10}{CD}$ and CD=$\frac{50}{3}$, and AD=15*(5/4)= $\frac{75}{4}$, so the

perimeter is $\frac{25}{2}+\frac{50}{3}+\frac{75}{4}=\frac{150}{12}+\frac{200}{12}+\frac{225}{12}=\frac{575}{12}$