FAIRFIELD COUNTY MATH LEAGUE 2023–2024
Match 2
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Factors and Multiples

1-1 How many positive integers n, $2 \leq n \leq 50$, have at most two prime factors? (Recall that 1 is not prime.)
[Answer: 47]

There are 49 total integers in the range provided in the problem. Of those, only two—30 $(2 * 3 * 5)$ and 42 $(2 * 3 * 7)$--have more than two prime factors. Therefore the desired quantity is 47.

1-2 What is the smallest positive integer that has the same number of factors as 160?
[Answer: 60]

The number 160 has a prime factorization of $2^5 * 5$, giving it $6 * 2 = 12$ total factors. A number will have 12 factors if it has a prime factorization on the form of a^{11}, $a^5 b$, $a^3 b^2$, or $a^2 bc$. Putting in the least possible values (2, 3, or 5) shows that the smallest possible value is $2^2 * 3 * 5 = 60$.

1-3 Let a, b, and c be integers greater than 1 such that $gcf(a, b) = 4$, $lcm(a, b) = 24$, and $gcf(ab, c) = 1$. What is the smallest possible value of $lcm(ab, c)$?
[Answer 480]

We can use the fact that $ab = gcf(a, b) * lcm(a, b)$, so $ab = 96$. Since $gcf(ab, c) = 1$, we know that c does not share any factors with a or b and that $lcm(ab, c) = abc$. Since $lcm(a, b) = 24$, we know that ab has prime factors of 2 and 3. Therefore, the smallest possible value of c is 5 and the desired value is $96 * 5 = 480$.

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Round 2: Polynomials and Factoring

2-1 Find the sum of all positive values of c such that the expression x^2+7x+c is factorable into two binomials with integer coefficients.
[Answer: 28]

Because c is positive, 7 must be the sum of the two positive factors that make c. Therefore c can be $6*1=6, 5*2=10$, or $4*3=12$, so the desired value is $6+10+12=28$.

2-2 Let a be the larger zero of $f(x)=x^2-11x+24$, and let b be the largest integer such that $g(x)=x^2+ax+b$ has two real irrational zeros. Find $f(b)$.
[Answer: 66]

Because $f(x)$ factors into $f(x)=(x-3)(x-8)$, the larger zero is $a=8$. Then the largest value b is required such that 8^2-4b is a positive number that is not a perfect square. When $b=16$ the quantity is 0, and when $b=15$ the quantity is 4, but when $b=14$ the quanti=ty is 8, so $b=14$. Therefore the desired quantity is $14^2-11(14)+24=196-154+24=66$.

2-3 The polynomial $f(x)=2x^3+4x^2+px-6$, where p is an integer, has at least one real rational zero. If A is the greatest possible value of p and B is the least possible value of p, find the value of $A-B$
[Answer: 95]

If the polynomial has a rational zero, it must be of the form $\pm\frac{\{1,2,3,6\}}{\{1,2\}}$, or $\pm\left\{1,2,3,6,\frac{1}{2},\frac{3}{2}\right\}$. We know that for any zero x of the polynomial, $p=-\frac{2x^3+4x^2-6}{x}=-2x^2-4x+\frac{6}{x}$. Note that all non-integer rational zeros would produce a non-integer value of p. Of the remaining possibilities, the value of p is maximized when $x=1$, producing a value of $p=-2(1)^2-4(1)+\frac{6}{1}=0$. The value of p is minimized when $x=6$, producing a value of $p=-2(6)^2-4(6)+\frac{6}{6}=-95$. Therefore the desired quantity is $0-(-95)=95$.

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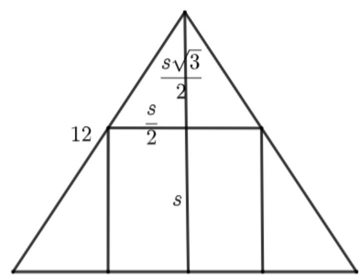
Round 3: Area and Perimeter

3-1 If a square's area is ten times its perimeter, what is its perimeter?
[Answer: 160]

If x is the side length of the square, then $x^2 = (10)(4x)$, which gives $x = 40$. Therefore the desired quantity is $4(40) = 160$.

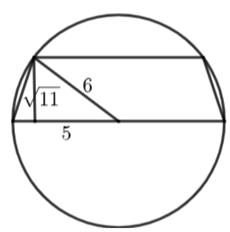
3-2 A square is inscribed in an equilateral triangle with perimeter 36. The square has a side length of $a\sqrt{b} - c$ where a, b, and c are positive integers and b has no perfect square factors greater than 1. Find $a + b + c$.
[Answer: 63]

See the diagram. Let the length of one side of the square be s. Using similar triangles, we can determine that the altitude of the equilateral triangle (with side lengths 12) is equal to $s + \frac{\sqrt{3}}{2}s$, and this equals a length of $6\sqrt{3}$. Therefore $s = \frac{6\sqrt{3}}{1+\frac{\sqrt{3}}{2}}$, which when rationalized becomes $12\sqrt{3}(2 - \sqrt{3}) = 24\sqrt{3} - 36$, making the desired quantity $24 + 3 + 36 = 63$.



3-3 An isosceles trapezoid is inscribed in a circle with area 36π such that the longer base of the trapezoid is a diameter of the circle. If the trapezoid has height $\sqrt{11}$, then its perimeter is $a + b\sqrt{c}$, where a, b, and c are positive integers and c has no perfect square factors greater than 1. Find $a + b + c$.
[Answer: 29]

See the diagram. Drawing in the radii from each endpoint of the smaller base to the center of the larger base of the trapezoid, we find that half the length of the smaller base is $\sqrt{36 - 11} = 5$. This means the smaller base has length 10 and each lateral side length is $\sqrt{1 + 11} = 2\sqrt{3}$. This make the perimeter $22 + 4\sqrt{3}$, making the desired quantity $22 + 4 + 3 = 29$.



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Round 4: Absolute Value & Inequalities

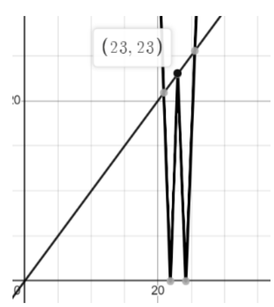
4-1 Evaluate the expression: $\left|5 - |5^2 - 5^3|\right|$
[Answer: 95]

Evaluating the exponents gives $|5 - |25 - 125||$, yielding $|5 - 100| = 95$.

4-2 Consider the equation $|ax - 8| = b$, where a and b are positive integer constants less than 100. If this equation has two solutions for x, x_1 and x_2, and $|x_1 - x_2| = \frac{3}{2}$, find the number of ordered pairs (a, b).
[Answer: 24]

Assume $x_1 > x_2$. Then $ax_1 - 8 = b$, yielding $x_1 = \frac{b+8}{a}$, and $ax_2 - 8 = -b$, yielding $x_2 = \frac{-b+8}{a}$. This means $x_1 - x_2 = \frac{2b}{a} = \frac{3}{2}$, so $a = \frac{4}{3}b$. Therefore the smallest positive integer values of a and b that work are (4,3) and the largest values less than 100 that work are $(96, 72)$, making a total of 24 ordered pairs.

4-3 The graph of the function $f(x) = mx$, where m is a positive constant, intersects the graph of the function $g(x) = |x - 20|x - 23||$ exactly three times. The largest x −coordinate of one of the points of intersection is $\frac{p}{q}$, where p and q are relatively prime integers. Find $p + q$.
[Answer: 239]



See the diagram. To visualize the graph of $g(x)$, consider an absolute value function with a vertex at (23,0), stretched vertically by a factor of 20, reflected over the x −axis, then shifted up by x. This produces an absolute value function with a vertex (which is its maximum) at (23,23), which is also a piecewise function of $x - 20(23 - x)$ or $21x - 460$ for $x < 23$ and $x - 20(x - 23)$ or $-19x + 460$ for $x > 23$. Finally we take the absolute value of this function, producing a "W" shaped graph. A line going through the origin will intersect this function exactly three times only if one of the intersections is the vertex located at (23,23). Therefore $m = 1$. The greatest point of intersection between $f(x) = x$ and $g(x)$ will be where $f(x)$ intersects the reflected portion of $g(x)$ to the right of the vertex, or $19x - 460$. Therefore setting $x = 19x - 460$, we get $x = \frac{460}{18} = \frac{230}{9}$, so the desired quantity is $230 + 9 = 239$.

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<u>Round 5: Law of Sines and Cosines</u>

5-1 In triangle ABC, $AB = 3(BC)$ and $m\angle B = 60°$. Find the value of $\left(\frac{AC}{BC}\right)^2$.
[Answer: 7]

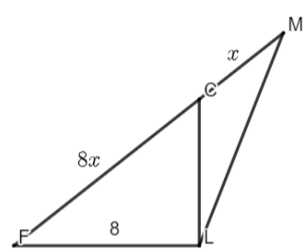
Since $(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos(60°)$, we have $(AC)^2 = 9(BC)^2 + (BC)^2 - 3(BC)^2$, so $(AC)^2 = 7(BC)^2$, making the desired quanity $\frac{(AC)^2}{(BC)^2} = 7$.

5-2 Consider triangle ABC, where $AB = 5$, $BC = 6$, and $\tan(B) = 2$. $(AC)^2 = p - q\sqrt{r}$, where p, q, and r are positive integers and r has no perfect square factors greater than 1. Find $p + q + r$.
[Answer: 78]

Because $\tan(B) > 0$, we know that angle B is acute and so $\cos(B) > 0$. We also know that $\tan^2(B) = \frac{1-\cos^2(B)}{\cos^2(B)} = 4$, so $\cos^2(B) = \frac{1}{5}$ and therefore $\cos(B) = \frac{\sqrt{5}}{5}$. Using the law of cosines, we have $(AC)^2 = 5^2 + 6^2 - 2(5)(6)\left(\frac{\sqrt{5}}{5}\right) = 61 - 12\sqrt{5}$, making the desired quantity $61 + 12 + 5 = 78$.

5-3 Consider triangle FML with obtuse angle L. $FL = 8$ and the area of FML is 48. Point C lies on $\overline{FM}$ such that $\overline{FL} \perp \overline{CL}$ and $FC = 8CM$. Find FM.
[Answer: 15]

See the diagram. Let $CM = x$, and so $FC = 8x$ and $FM = 9x$. This also means from triangle FLC that $LC = 8x\sin(F)$. We know $\frac{1}{2}(9x)(8)\sin(F) = 48$. Therefore $8x\sin(F) = \frac{32}{3} = LC$, and we can solve for x using $8^2 + \left(\frac{32}{3}\right)^2 = 64x^2$ to get $x = \frac{5}{3}$ (or note that $\frac{24}{3}, \frac{32}{3}$, and $\frac{40}{3}$ is a Pythagorean triple). Therefore $FM = 9\left(\frac{5}{3}\right) = 15$.



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Round 6: Equations of Lines

6-1 A line with equation $3x - 8y = C$, where C is a constant, contains the point $(24, 20)$. What is the y-coordinate of the y-intercept?
[Answer: 11]

Substituting the point $(24,20)$ into the equation yields $3(24) - 8(20) = -88$. Therefore the desired quantity can be found by solving $3(0) - 8y = -88$, yielding a value of 11.

6-2 Line l_1 has a slope of $\frac{5}{3}$ and a y −intercept of $(0, b)$, where b is a positive integer. Line l_1 is reflected across the x −axis to make line l_2, and the two lines intersect at $x = -21$. What is the value of b?
[Answer 35]

Note that l_1 and l_2 will intersect on the x −axis. Therefore the value of b can be found by substituting $(-21,0)$ into the equation $y = \frac{5}{3}x + b$, and hence $0 = \frac{5}{3}(-21) + b$, yielding $b = 35$.

6-3 A line with equation $y = mx$, where m is a positive constant, has the property that decreasing the slope by 95% would reduce the measure of the angle made between the line and the x −axis in the first quadrant by 50%. Find the value of m^2.
[Answer: 360]

One way to solve this is to use double-angle formulas. Let the original slope be m and the new reduced slope be m_0. Using the tangent double-angle formula, we get $m = \frac{2m_0}{1-m_0^2}$. We also know that $m_0 = \frac{1}{20}m$. Therefore we have $m = \frac{\frac{1}{10}m}{1-\frac{1}{400}m^2}$, or $m = \frac{40m}{400-m^2}$, giving $400 - m^2 = 40$, leading to a final answer of $m^2 = 360$.

FAIRFIELD COUNTY MATH LEAGUE 2023-2024
Match 2
Team Round

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1. The function $floor(x)$, also known as the greatest integer function, maps x to the greatest integer that is less than or equal to x. Consider the five-digit number $3abcd$, where the last four digits $a, b, c,$ and d are unknown. This number has the property that $floor(\frac{3abcd}{8}) = abcd$ (a four-digit number comprised of the same unknown four digits in order). What is the four-digit number represented by $abcd$?
[Answer: 4285]

One way to solve this problem is to set up the division and derive the solution one digit at a time. In other words, first consider that $floor\left(\frac{34}{8}\right) = 4$, and 4 is the only digit that will have this property. Therefore $a = 4$. Continuing the division leads us to consider $floor\left(\frac{2b}{8}\right) = b$, and $b = 2$ is the only digit that works. Subsequently we get $c = 8$ and $d = 5$, making the solution 4285.

2. The polynomial $x^4 + kx + 35$, where k is a positive integer, is factorable into two quadratic trinomial factors with integer coefficients. What is the value of k?
[Answer: 204]

Consider that $x^4 + kx + 35 = (x^2 + ax + b)(x^2 + cx + d) = x^4 + (a + c)x^3 + (b + d + ac)x^2 + (ad + bc)x + bd$. We see that $c = -a$, so we have $x^4 + (b + d - a^2)x + a(d - b)x + bd$. Therefore b and d are integers that multiply to make 35 and add to a perfect square. Since 35 has only two pairs of factors, 1*35 or 7*5, we see that the pair of factors must be 1 and 35 since $1 + 35 = 36$, meaning $a = 6$. Therefore the desired quantity $k = 6(35 - 1) = 204$.

3. A rectangle has the property that its dimensions are integers and its area and perimeter are equal. Find the sum of all possible areas of the rectangle.
[Answer: 34]

Setting up $ab = 2a + 2b$, we isolate one variable to get $a = \frac{2b}{b-2}$. Since $b > 2$ (and $a > 2$ also), first trying $b = 3$ yields $a = 6$ for an area of 18. Letting $b = 4$ yields $a = 4$ for an area of 16. For greater values of b the values reverse so no new integer ordered pairs will come from the equation. Therefore the desired quantity is $18 + 16 = 34$.

4. The figure enclosed on the xy-plane by the equation $|x + y| + 3|x - y| < 8$ has an area of $\frac{a}{b}$, where a and b are positive integers with no common factors greater than 1. Find $a - b$.
[Answer: 61]

This inequation will produce a rhombus on the coordinate plane. One diagonal will be along the line $y = x$ and the vertices will occur when $|x + y| = 8$, giving coordinates of $(4,4)$ and $(-4,-4)$. The other diagonal is along the line $y = -x$ and the vertices will occur when $|x - y| = \frac{8}{3}$, giving coordinates of $\left(\frac{4}{3}, -\frac{4}{3}\right)$ and $\left(-\frac{4}{3}, \frac{4}{3}\right)$. The easiest way to find the area is to compute $\frac{1}{2}d_1d_2$, or $\frac{1}{2}\left(8\sqrt{2}\right)\left(\frac{8}{3}\sqrt{2}\right) = \frac{64}{3}$, making the desired quantity $64 - 3 = 61$.

5. Two spotlights on level ground (assume elevation of 0) are aimed at a tightrope performer who stands on a rope that is stretched directly above the pathway between the lights. Light A makes angle A with the ground and light B makes angle B with the ground, and it is known that angle B is twice the measure of angle A. If the performer is 500 feet from light A and 350 feet from light B, then the height of the performer above the ground in feet is $\frac{a\sqrt{b}}{c}$, where $a, b,$ and c are positive integers, a and c have no common factors greater than 1 and b has no perfect square factors greater than 1. Find $ab + c$.
[Answer: 6007]

Using the law of sines, we get $\frac{350}{\sin(A)} = \frac{500}{\sin(B)}$, and since $B = 2A$, we have $\frac{350}{\sin(A)} = \frac{250}{\sin(A)\cos(A)}$, yielding $\cos(A) = \frac{250}{350} = \frac{5}{7}$. Therefore $\sin(A) = \frac{2\sqrt{6}}{7} = \frac{h}{500}$ where h is the height above ground. Hence $h = \frac{1000\sqrt{6}}{7}$, making the desired quantity $(1000)(6) + 7 = 6007$.

6. The parametric equations $x = \frac{3}{t-6} + 2$ and $y = \frac{2}{t-6} - 4$ produce a line on the xy −plane with a discontinuity at the point (a, b). Line l is perpendicular to this line and contains the point (a, b) and $(-12, c)$. What is the value of c?
[Answer: 17]

Looking at the parametric equations, we see that $x \neq 2$ and $y \neq -4$, so the line has a discontinuity at $(2, -4)$. Also note that $y = \frac{2}{3}x + b$ for some y-intercept b, so the slope of the original line is $\frac{2}{3}$. This means that line l has a slope $-\frac{3}{2}$. We can solve for c using $\frac{c+4}{-12-2} = -\frac{3}{2}$, which produces the desired quantity of $c = 17$.